Set Theory: Laws and Proofs

Ian Ludden

Learning Objectives

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- Apply definitions and laws to set theoretic proofs.

Set Theory Properties/Identities/Laws

$$A \overset{\cup}{\cancel{\times}} B = B \overset{\cup}{\cancel{\times}} A$$

$$\{s \in U : \underline{s \in A} \overset{\text{or}}{\cancel{\text{and}}} \underline{s \in B}\}$$

- Commutative, associative

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

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- Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Theory Properties/Identities/Laws

- Commutative, associative
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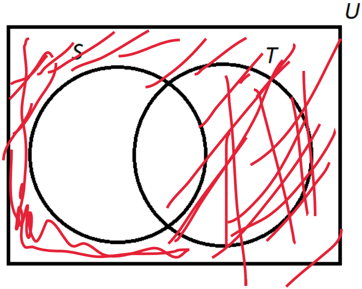
- Double complement

$\overline{(\bar{A})} = A$



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- Distributive
- Double complement
- De Morgan's Laws:

$\neg(p \wedge q) = \neg p \vee \neg q$

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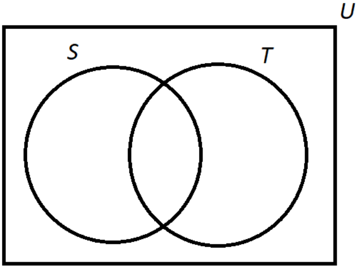

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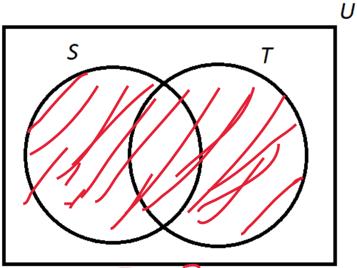

$\overline{S \cup T} = \{x \in U : \neg(x \in S \text{ or } x \in T)\}$

$x \notin S$ and $x \notin T$

$x \in \overline{S}$ $\qquad$ $x \in \overline{T}$

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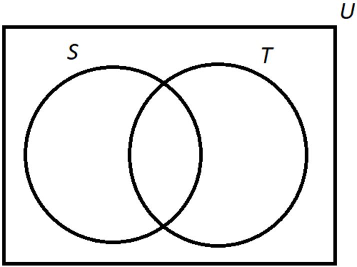
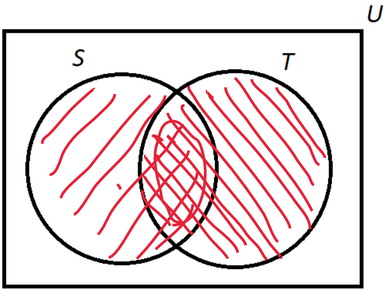
  - $\overline{S \cup T} = \overline{S} \cap \overline{T}$
- And many more...



Cardinality after Set Operations



- Size of set union

$|S \cup T| = ?$ $\qquad |S| + |T| - |S \cap T|$

- Size of set union
- Size of Cartesian product (***product rule***)

Menu

Appetizer	Entree	Dessert
Wings	Pizza	Gelato
Mozz. sticks	Pasta	Rhubarb Pie
Onion rings	Steak	Choc. cake
Salad	Chicken	Cheesecake
Calamari		Cookie
Soup		

$M = A \times E \times D$

$= \{(a, e, d) : a \in A, e \in E, d \in D\}$

$|M| = |A \times E \times D| = |A| \cdot |E| \cdot |D|$

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- $A \subseteq B \longleftrightarrow \forall a \in A, a \in B$
- Let $a \in A$ be arbitrary.
- [Details] don't use any facts about 'a' other than $a \in A$.
- So $a \in B$. Since a was arbitrarily chosen, we conclude $A \subseteq B$. $\square$

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Example

Define $A = \{a \in \mathbb{Z} : a^2 - 9 \text{ is odd and } |a| < 25\}$ and $B = \{b \in \mathbb{Z} : b \text{ is even}\}$. Prove $A \subseteq B$.

Let $a \in A$ be arbitrary. Then, $a^2 - 9 = (a-3)(a+3)$ is odd and $|a| < 25$.

Since $(a-3)(a+3)$ is odd, $a-3$ is odd and $a+3$ is odd.

By defn, $a - 3 = 2k+1$ for some $k \in \mathbb{Z}$.

$a = 2k+4 = 2(k+2)$. Hence a is even, so $a \in B$. $\square$

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To prove set equality, show inclusion in both directions

$A = B \longleftrightarrow A \subseteq B$ and $B \subseteq A$.

Another Set Proof

Let $A, B, C \subseteq U$. Prove that $(A - B) \subseteq C$ if and only if $(A - C) \subseteq B$.

iff

($\rightarrow$) Suppose $(A-B) \subseteq C$. (1)

(W.T.S. $(A-C) \subseteq B$.)

Let $a \in (A-C)$ be arb. Then $a \in A$, and $a \notin C$.

Since $a \notin C$, $a \notin (A-B)$ by assumption (1)

Then $a \in A-(A-B) = A \cap B \subseteq B$.

Hence $a \in B$, and we conclude $(A-C) \subseteq B$.

($\leftarrow$) (Similar).

$\square$

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Summary of set theory laws:
https://en.wikipedia.org/wiki/Algebra_of_sets