

Number Theory: The Euclidean Algorithm

Ian Ludden

Learning Objectives

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← "divides"

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- $\text{gcd}(8, 12) = 4$
- $\text{gcd}(-35, 20) = 5$
- $\text{gcd}(a, b) = \text{gcd}(b, a)$
- For any integer $a \neq 0$, $\text{gcd}(a, 0) = |a|$

$$\forall n \in \mathbb{Z}, n \mid 0.$$

$$0 = n \cdot 0.$$

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- $\gcd(-35, 20) = 5$
- $\gcd(a, b) = \gcd(b, a)$
- For any integer $a \neq 0$, $\gcd(a, 0) = |a|$
- $\gcd(0, 0)$ is undefined

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Theorem

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Examples

- $a = 4, b = 7$

$$\begin{aligned} \text{gcd}(4, 7) &= 1 && \text{"relatively prime"} \\ \text{lcm}(4, 7) &= \frac{4 \cdot 7}{1} = 28 \end{aligned}$$

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Examples

- $a = 4, b = 7$

- $a = 20, b = 12$

$$\text{gcd}(20, 12) = 4$$

$$\text{lcm}(20, 12) = \frac{20 \cdot 12}{4} = 60.$$

- By comparing prime factorizations (slow)

$$a = 168, \quad b = 228$$

$$a = 2^3 \cdot 3^1 \cdot 7^1 \cdot 1^0$$

$$b = 2^2 \cdot 3^1 \cdot 7^1 \cdot 1^0$$

$$\begin{aligned} \gcd(a, b) &= 2^2 \cdot 3^1 \cdot \cancel{7^1} \cdot \cancel{1^0} \\ &= 12 \end{aligned}$$

Computing gcd

- By comparing prime factorizations (slow)
- By the Euclidean algorithm (fast, easy to do by hand)

Theorem

For any integers a and b , where $b > 0$, there exist a unique quotient $q \in \mathbb{Z}$ and remainder $r \in \mathbb{Z}$ such that

- 1 $a = bq + r$ and
- 2 $0 \leq r < b$.

The Division Algorithm, Revisited

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Claim

For any integers a , b , q , and r , with b positive, if $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.

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See textbook, Section 4.6, for proof of claim

$$\underbrace{n|a \text{ and } n|b}_{\text{handwritten}} \Rightarrow n|(bq) \Rightarrow n|(a - bq) \Rightarrow n|r$$

The Euclidean algorithm

Repeatedly apply the division algorithm and the claim

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"function"



```
procedure gcd(a, b)
```

```
  r := remainder(a, b)
```

```
  if r == 0
```

```
    return b
```

```
  else
```

```
    return gcd(b, r)
```

$$a \div b$$

$$a = bq + r$$

$$a = bq + 0$$

$$\gcd(a, b) \leq a$$

$$\gcd(a, b) \leq b$$

recursion



The Euclidean algorithm

Repeatedly apply the division algorithm and the claim

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procedure gcd(a, b)
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Example

$a = 168, b = 456$

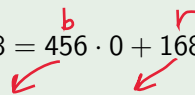
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Example

$$a = 168, b = 456$$

$$168 = 456 \cdot 0 + 168$$


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Example

$$a = 168, b = 456$$

$$168 = 456 \cdot 0 + 168$$

$$456 = 168 \cdot 2 + 120$$

a ← b ← a ← r

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$$456 = 168 \cdot 2 + 120$$

$$168 = 120 \cdot 1 + 48$$


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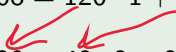
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$$a = 168, b = 456$$

$$168 = 456 \cdot 0 + 168$$

$$456 = 168 \cdot 2 + 120$$

$$168 = 120 \cdot 1 + 48$$

$$120 = 48 \cdot 2 + 24$$


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Example

$$a = 168, b = 456$$

$$168 = 456 \cdot 0 + 168$$

$$456 = 168 \cdot 2 + 120$$

$$168 = 120 \cdot 1 + 48$$

$$120 = 48 \cdot 2 + 24$$

$$48 = 24 \cdot 2 + 0$$

Recap: Learning Objectives

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