Number Theory: The Euclidean Algorithm

Ian Ludden

Learning Objectives

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- $\gcd(a, b) = \gcd(b, a)$
- For any integer $a \neq 0$, $\gcd(a, 0) = |a|$
- $\gcd(0, 0)$ is undefined

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Examples

- $a = 4$, $b = 7$
- $a = 20$, $b = 12$

Computing gcd

- By comparing prime factorizations (slow)

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- By the Euclidean algorithm (fast, easy to do by hand)

The Division Algorithm, Revisited

Theorem

For any integers a *and* b*, where* $b > 0$*, there exist a unique quotient* $q \in \mathbb{Z}$ *and remainder* $r \in \mathbb{Z}$ *such that*

1. $a = bq + r$ *and*
2. $0 \leq r < b$*.*

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Claim

For any integers a, b, q, and r, with b positive, if $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.

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See textbook, Section 4.6, for proof of claim

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Repeatedly apply the division algorithm and the claim

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procedure gcd(a, b)
    r := remainder(a, b)
    if r == 0
        return b
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        return gcd(b, r)
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Example

$a = 168$, $b = 456$

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Example

$a = 168, b = 456$

$$168 = 456 \cdot 0 + 168$$

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Example

$a = 168$, $b = 456$

$$168 = 456 \cdot 0 + 168$$

$$456 = 168 \cdot 2 + 120$$

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$456 = 168 \cdot 2 + 120$

$168 = 120 \cdot 1 + 48$

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$168 = 120 \cdot 1 + 48$

$120 = 48 \cdot 2 + 24$

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$a = 168$, $b = 456$

$$168 = 456 \cdot 0 + 168$$

$$456 = 168 \cdot 2 + 120$$

$$168 = 120 \cdot 1 + 48$$

$$120 = 48 \cdot 2 + 24$$

$$48 = 24 \cdot 2 + 0$$

Recap: Learning Objectives

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