Introduction to Number Theory

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Learning Objectives

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- Recall basic definitions from number theory.

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- Recall basic definitions from number theory.
- Apply the definition of “divides” in direct proofs.
- State the Division Algorithm theorem.

What is number theory?

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Definition

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Quote

"Mathematics is the queen of the sciences, and number theory is the queen of mathematics." – Gauss

Motivation

Number theory applications:

Motivation

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- RSA Encryption

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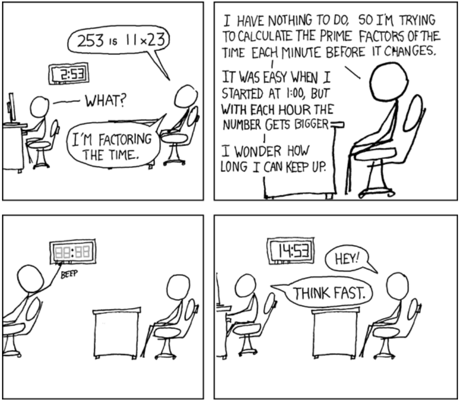

Source: `https://xkcd.com/247/`

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Examples

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Examples

- 0 is even, because $0 = 2 \cdot 0$
- 173 is odd, because $173 = 2 \cdot 86 + 1$

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Examples

- 0 is even, because $0 = 2 \cdot 0$
- 173 is odd, because $173 = 2 \cdot 86 + 1$
- -128 is even, because $-128 = 2(-64)$

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Examples

- $3 \mid 6$ Proof: $n = 2$. $6 = 3 \cdot 2$

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Examples

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- $6 \nmid 3$ $\quad \nexists n \in \mathbb{Z} \;\; 3 = 6n$.

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Examples

- $3 \mid 6$
- $6 \nmid 3$
- $51 \mid 0$ $n = 0$ $0 = 51 \cdot 0$

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Examples

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- $6 \nmid 3$
- $51 \mid 0$
- $0 \mid 0$

$\cancel{\frac{0}{0}}$

$n = 0$

$0 = 0 \cdot 0$

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Examples

- $3 \mid 6$
- $6 \nmid 3$
- $51 \mid 0$
- $0 \mid 0$
- $(-5) \mid 30$

$n = -6$

$30 = (-5)(-6)$

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Examples

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- $6 \nmid 3$
- $51 \mid 0$
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- $(-5) \mid 30$
- $11 \mid (-121)$ $n = -11$

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Warning

Tempting: "a divides b if $\frac{b}{a}$ is an integer." ***Don't do this.*** Breaks for $a = 0$; also, see Section 4.3 of the textbook.

Example

Claim: For all integers m and n, if m is even and $m \mid n$, then n is even.

Proof: Let m and n be arbitrary integers.

Suppose m is even and $m|n$.

By defn., $m = 2k$ for some $k \in \mathbb{Z}$.

By defn., $n = m \cdot p$ for some $p \in \mathbb{Z}$.

Then,

$$\begin{aligned} n &= m \cdot p \\ &= 2k \cdot p \\ &= 2(kp) \\ &= 2q, \end{aligned}$$

where $q = kp \in \mathbb{Z}$.

Hence n is even

$\square$

Direct proofs using divides

Example

Claim: $\forall p, q, r \in \mathbb{Z}$, $(p \mid q) \wedge (q \mid r) \rightarrow (p \mid r)$.
(Transitive property of divides.)

Proof: Let $p, q, r \in \mathbb{Z}$ be arb.

Suppose $(p \mid q) \wedge (q \mid r)$. By defn., $\exists a, b \in \mathbb{Z}$ such that

$q = p \cdot a$ and $r = q \cdot b$.

Then,

$$r = q \cdot b$$
$$= p \cdot a \cdot b$$

$r = p \cdot k$, where $k = ab \in \mathbb{Z}$.

Hence $p \mid r$. $\square$

The Division Algorithm

The Division Algorithm

Theorem

For any integers a *and* b*, where* $b > 0$*, there exist a unique quotient* $q \in \mathbb{Z}$ *and remainder* $r \in \mathbb{Z}$ *such that*

1. $a = bq + r$ *and*
2. $0 \leq r < b$.

$q = a \mathbin{//} b$ integer division

$r = a \% b$

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Examples

- $a = 173$, $b = 5$ $q = 34$, $r = 3$ $173 = 5(34) + 3$

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Examples

- $a = 173,\ b = 5$
- $a = -20,\ b = 4$ $\quad q = -5,\ r = 0 \quad -20 = 4(-5) + 0$

$a = -19,\ b = 4 \quad q = -5,\ r = 1 \quad -19 = 4(-5) + 1$

~~$-19 = 4(-4) + (-3)$~~

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1. $a = bq + r$ *and*
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Examples

- $a = 173$, $b = 5$
- $a = -20$, $b = 4$
- $a = 12$, $b = 97$ $q=0, r=12$ $12 = 97(0) + 12$

Recap: Learning Objectives

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