Introduction to Number Theory

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Learning Objectives

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- Apply the definition of “divides” in direct proofs.
- State the Division Algorithm theorem.

What is number theory?

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Definition

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Quote

"Mathematics is the queen of the sciences, and number theory is the queen of mathematics." – Gauss

Motivation

Number theory applications:

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- RSA Encryption

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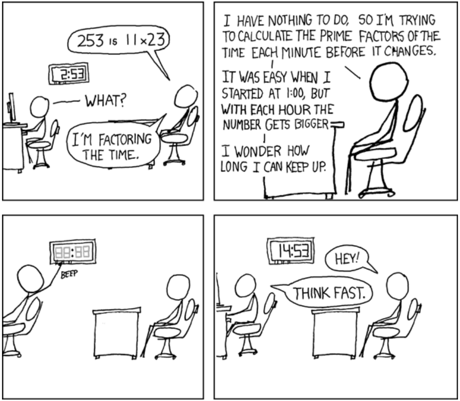

Source: `https://xkcd.com/247/`

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Examples

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Examples

- 0 is even, because $0 = 2 \cdot 0$
- 173 is odd, because $173 = 2 \cdot 86 + 1$

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Examples

- 0 is even, because $0 = 2 \cdot 0$
- 173 is odd, because $173 = 2 \cdot 86 + 1$
- -128 is even, because $-128 = 2(-64)$

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- $51 \mid 0$
- $0 \mid 0$
- $(-5) \mid 30$

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Warning

Tempting: "a divides b if $\frac{b}{a}$ is an integer." ***Don't do this.*** Breaks for $a = 0$; also, see Section 4.3 of the textbook.

Direct proofs using divides

Example

Claim: For all integers m and n, if m is even and $m \mid n$, then n is even.

Direct proofs using divides

Example

Claim: $\forall p, q, r \in \mathbb{Z}, (p \mid q) \wedge (q \mid r) \rightarrow (p \mid r)$.
(Transitive property of divides.)

The Division Algorithm

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Theorem

For any integers a *and* b*, where* $b > 0$*, there exist a unique quotient* $q \in \mathbb{Z}$ *and remainder* $r \in \mathbb{Z}$ *such that*

1. $a = bq + r$ *and*
2. $0 \leq r < b$.

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- $a = 173$, $b = 5$

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- $a = -20$, $b = 4$

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Examples

- $a = 173$, $b = 5$
- $a = -20$, $b = 4$
- $a = 12$, $b = 97$

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- Recall basic definitions from number theory.
- Apply the definition of “divides” in direct proofs.
- State the Division Algorithm theorem.