Proving Universal Claims

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Learning Objectives

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- Prove universal claims directly.
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- Prove universal claims by splitting into cases.

Direct proof: The hard/impossible way

Goal: Prove $\forall x \in X, p(x)$.

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Strategy: Individually show the predicate is true for each element in the universe.

$\forall n \in \mathbb{N},\ n^2 - 4n + 5 \geq 0.$

Consider $n = 0$. $0^2 - 4(0) + 5 = 5 \geq 0.$ ✓

$n = 1$. ✓

$n = 2$

$\vdots$

Direct proof: The more reasonable way

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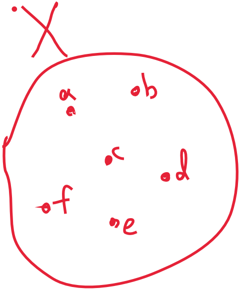
Strategy: Pick an *arbitrary* element in the universe, and show the predicate is true for that element.



Let x be an arbitrary element of X.

[details]

p(x).

Direct proof: The more reasonable way

Example: Prove $\forall n \in \mathbb{N}, n^2 - 4n + 5 \geq 0$.

Direct proof: The more reasonable way

Example: Prove $\forall n \in \mathbb{N}, n^2 - 4n + 5 \geq 0$.

Proof: Let n be an arbitrary natural number.

Then,

$$
\begin{aligned}
n^2 - 4n + 5 &= n^2 - 4n + 4 + 1 \\
&= (n-2)(n-2) + 1 \\
&= (n-2)^2 + 1 \\
&\geq 0 + 1 \\
&\geq 0. \quad \square
\end{aligned}
$$

Proof by contrapositive

$p \to q$

$\neg q \to \neg p$

Goal: Prove $\forall x \in X, p(x) \to q(x)$.

Let $x \in X$ be arbitrary.

Suppose $p(x)$.

⋮ ???

Then $q(x)$.

Proof by contrapositive

Goal: Prove $\forall x \in X, p(x) \to q(x)$.

Equivalent: Prove $\forall x \in X, \neg q(x) \to \neg p(x)$.

Proof by contrapositive

$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$ De Morgan's law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example

Prove: $\forall x, y \in \mathbb{R}, x + y \geq 0 \rightarrow (x \geq 0) \vee (y \geq 0)$.

Pf.: We will prove the contrapositive of the claim, that is,

$$\forall x, y \in \mathbb{R}, \; x < 0 \wedge y < 0 \rightarrow x + y < 0.$$

Let x and y be arbitrary real numbers.

Suppose $x < 0$ and $y < 0$.

Then,

$$x + y < x + 0 \quad (\text{since } y < 0)$$
$$< 0. \quad (\text{since } x < 0)$$

So $x + y < 0$. $\square$

Proof by cases

Goal: Prove $\forall x \in X, p(x)$.

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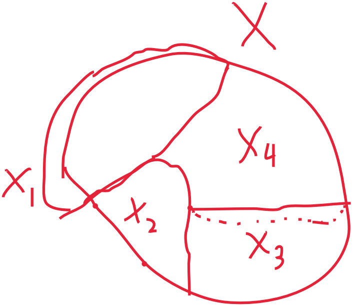
Equivalent:

1. Find sets $X_1, X_2, \ldots, X_k$ such that $X = X_1 \cup X_2 \cup \ldots \cup X_k$.
   (It's ok if your sets overlap.)



Proof by cases

Goal: Prove $\forall x \in X, p(x)$.

Equivalent:

1. Find sets $X_1, X_2, \ldots, X_k$ such that $X = X_1 \cup X_2 \cup \ldots \cup X_k$.
   (It's ok if your sets overlap.)
2. Prove each case, that is, for $i = 1$ to k, prove $\forall x \in X_i, p(x)$.

Proof by cases

Example 1

Claim: For every integer n, $n^3 \geq 0$ or $n \geq 2n$.

Proof: Let n be an arbitrary integer.

The proof is by case analysis. There are two cases: $n \geq 0$, or $n \leq 0$.

Case 1: Suppose $n \geq 0$.

Then, $n^3 = n \cdot n^2$.

Since $n \in \mathbb{Z}$, $n^2 \geq 0$.

so n^3 is the product of two nonnegative integers, which is nonneg.

Case 2: Suppose $n \leq 0$.

Then,

$$n = n + 0$$
$$\geq n + n \quad (\text{since } n \leq 0)$$
$$= 2n,$$

The claim holds in Case 1 and Case 2, so the claim holds for all integers $\square$

Proof by cases

Example 2

Claim: Every set of four people includes a trio of friends or a pair of strangers.

Pf: Let S be an arb. set of 4 people.
Let x be an arb. person in S.
The proof is by cases. There are two cases:
[] x knows all three other people, or
() ∃y ∈ S, y ≠ x, such that x has not met y.

Case 1: Suppose [], a, b, c.
This case splits into two subcases.
Case 1.1: Suppose some pair of a, b, c know each other. Then that pair with x is a trio of friends.
Case 1.2: Suppose some pair of a, b, c does not know each other.

Case 2: Suppose S contains a person y whom x has not met.
Then x and y form a pair of strangers.

The claim holds in both cases, so the claim holds for all sets of 4 people. □

Recap: Learning Objectives

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