Introduction to Proofs

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Learning Objectives

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- Disprove a universal claim with a concrete counterexample.

Negating statements with quantifiers

$$\neg [\forall x \; p(x)] \equiv \exists x \; \neg p(x)$$

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Example

Everybody loves Raymond.

$\forall p \in P, \; r(p).$ Negation: $\exists p \in P, \; \neg r(p).$

There exists a person who does not love Raymond.

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Example

There is an integer whose value is three less than that of its square.

$\exists n \in \mathbb{Z}, \; n = n^2 - 3.$

$\forall n \in \mathbb{Z}, \; n \neq n^2 - 3.$

Proving existential claims

$\exists x \in X, \ P(x).$

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Example

There is an integer whose value is ***six*** less than that of its square.

$\exists n \in \mathbb{Z},\ n = n^2 - 6.$

Proof: Consider $n=3$. The LHS is 3, and the RHS is $3^2-6=3$. Hence the claim holds for $n=3$. Q.E.D. $\square$

Scratchwork: $n = n^2 - 6$

$n^2 - n - 6 = 0 \quad (n-3)(n+2) = 0 \Rightarrow n = 3$ or $n = -2$

Disproving universal claims

Typical approach:

Disproving universal claims

Disprove: $\forall x, p(x)$. $\rightarrow$ Prove: $\neg[\forall x, p(x)] \equiv \exists x, \neg p(x)$

Typical approach:

1. Negate the statement to get an existential claim.

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Example

Every prime number can be expressed as the sum of exactly two prime numbers (not necessarily distinct).

$P := \{\text{prime numbers}\}$

$\forall p \in P,\ t(p).$

① Negation: $\exists p \in P,\ \neg t(p).$

② (Counter)example: Consider $p=2$. The smallest prime # is 2, so the smallest sum of two prime numbers is 4. This is greater than $p=2$, so $\neg t(2)$. Hence $p=2$ is a counterexample for the claim. □

Recap: Learning Objectives

By the end of this lesson, you will be able to:

- Negate statements with quantifiers.
- Prove an existential claim with a concrete example.
- Disprove a universal claim with a concrete counterexample.