

Introduction to Proofs

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Learning Objectives

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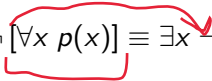
- Negate statements with quantifiers.
- Prove an existential claim with a concrete example.

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- Negate statements with quantifiers.
- Prove an existential claim with a concrete example.
- Disprove a universal claim with a concrete counterexample.

Negating statements with quantifiers

$$\neg [\forall x p(x)] \equiv \exists x \neg p(x)$$


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Example

Everybody loves Raymond.

$$\forall p \in P, r(p).$$

Negation: $\exists p \in P, \neg r(p).$

There exists a person who does not love Raymond.

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Example

There is an integer whose value is three less than that of its square.

$$\exists n \in \mathbb{Z}, n = n^2 - 3.$$

$$\forall n \in \mathbb{Z}, n \neq n^2 - 3.$$

Proving existential claims

$$\exists x \in X, p(x).$$

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- Give a concrete example (often easiest)

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- Non-constructive argument (sometimes useful; may see in future classes)

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Example

There is an integer whose value is **six** less than that of its square.

$$\exists n \in \mathbb{Z}, n = n^2 - 6.$$

Proof: Consider $n=3$.

The LHS is 3, and the RHS is $3^2 - 6 = 3$.

Hence the claim holds for $n=3$. Q.E.D. \square

Scratchwork: $n = n^2 - 6$

$$n^2 - n - 6 = 0 \quad (n-3)(n+2) = 0 \quad \Rightarrow \begin{matrix} n=3 \\ n=-2 \end{matrix}$$

Disproving universal claims

Typical approach:

Disproving universal claims

Dis Prove: $\forall x, p(x)$. \rightarrow Prove: $\neg[\forall x, p(x)] \equiv \exists x, \neg p(x)$

Typical approach:

- 1 Negate the statement to get an existential claim.

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- 2 Prove the existential claim with a concrete (counter)example.

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Example

Every prime number can be expressed as the sum of exactly two prime numbers (not necessarily distinct). $P := \{\text{prime numbers}\}$

$$\forall p \in P, \exists t(p).$$

① Negation: $\exists p \in P, \neg t(p).$

② (Counter)example: Consider $p=2$. The smallest prime $\neq 2$ is 3, so the smallest sum of two prime numbers is 4. This is greater than $p=2$, so $\neg t(2)$. Hence $p=2$ is a counterexample for the claim. \square

Recap: Learning Objectives

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