CS 173, Spring 2014 Midterm 2 Review Solutions

1. **Counting**

   (a) In our role-playing game, an evil character may be an elf or a troll, it may be red, green, brown, or black, and it may have scales or hair. A good character may be an elf or a human or a lion, it may be green, brown, or blue, and it has hair or fur. How many different character types do we have?

   Solution: There are $2 \cdot 4 \cdot 2 = 16$ types of evil characters. There are $3 \cdot 3 \cdot 2 = 18$ types of good characters. But there are 2 types of characters that could be good or evil. So we have a total of $16 + 18 - 2 = 32$ possible appearances.

   (b) Suppose we have a 26 character alphabet. How many 6-letter strings start with PRE or end with TH?

   Solution: There are 26^3 strings starting with PRE, 26^4 strings ending in TH, and 26 strings that start with PRE and end in TH. Thus we have a total of $26^3 + 26^4 - 26 = 26(26^2 + 26^3 - 1)$ total strings.

   (c) How many different 6-letter strings can I make out of the letters in the word "illini"?

   Solution: We calculate the number of permuations of 6 letters (6!) and divide out by the double-counting of the possibilities for l (2!) and for i (3!). This gives us $\frac{6!}{2!3!} = 5 \cdot 4 \cdot 3 = 60$ possible strings.

2. **Nested Quantifiers**

   Prove or disprove the statements in (a), (b), and (d). **Hint**: these proofs/disproofs are meant to be very brief.

   (a) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ \text{GCD}(x, y) = 1$

   Solution: True. Let $x = 1$. Note that $\text{GCD}(1, y) = 1$ for any choice of y since 1 divides all natural numbers (including 0).

   (b) $\forall x \in \mathbb{Z}^+,\ \exists y \in \mathbb{Z}^+,\ x = y^2$

   Solution: This says that all positive integers are perfect squares, which is false. Choose $x = 2$. If there were an integer y such that $2 = y^2$, then, $y = \sqrt{3}$ must be an integer, which is absurd.

   (c) Suppose that f is a function from $\mathbb{Z}_6$ to $\mathbb{Z}_8$, and $\exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$. Give a one sentence description of the function f.

   Solution: The function f sends all inputs to a single output $c \in \mathbb{Z}_8$, i.e., it is a constant function.

   (d) $\exists f : \mathbb{Z}_6 \to \mathbb{Z}_8,\ \exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$

   Solution: True. Let $c = [0]$ and simply take f to be the constant function which sends all inputs $x \in \mathbb{Z}_6$ to $[0] \in \mathbb{Z}_8$, i.e., $f(x) = [0]$ for all $x \in \mathbb{Z}_6$.

Function Proofs

(a) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is onto, then g is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

Solution: Suppose that $A = \{1, 2\}$, $B = \{3, 4, 5\}$, and $C = \{\text{red}, \text{blue}\}$. Define g by $g(1) = 3$ and $g(2) = 5$. Define f by $f(3) = \text{red}$, $f(4) = \text{red}$, and $f(5) = \text{blue}$.

Then $(f \circ g)(1) = \text{red}$ and $(f \circ g)(2) = \text{blue}$. So $f \circ g$ is onto because every element of C has a pre-image. However, g isn't onto because no element of A maps onto 4.

(b) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define the function $f : \mathbb{Z} \to \mathbb{Z}^2$ by $f(x) = (x^2, g(x))$. Prove that f is one-to-one.

Solution: Let x and y be integers. Suppose that $f(x) = f(y)$. By the definition of f, this means that $(x^2, g(x)) = (y^2, g(y))$. So then $x^2 = y^2$ and $g(x) = g(y)$. Since $g(x) = g(y)$ and g is one-to-one, $x = y$.

So we have that $f(x) = f(y)$ implies $x = y$. This means that f is one-to-one.

(c) Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n + 1) = 5f(n) - 6f(n - 1)$

Suppose we're proving that $f(n) = 3^n - 2^n$ for every positive integer n. State the inductive hypothesis and the conclusion of the inductive step.

Solution: Inductive hypothesis: suppose that $f(n) = 3^n - 2^n$ for $n = 1, 2, \ldots k$, for some integer k.

Conclusion of the inductive step: $f(k + 1) = 3^{k+1} - 2^{k+1}$.

Note 1: a strong hypothesis is required because the formula reaches back two integers.

Note 2: the variable k in the conclusion matches the upper bound in the hypothesis. A common mistake is to have it match the variable in the hypothesis equation (n). We're assuming that the equation holds for all values up through k, so we need to prove it holds for $k + 1$.

Induction

Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 1$

$f(1) = 6$

$\forall n \geq 2,\ f(n) = 6f(n-1) - 9f(n-2)$

Use strong induction on n to prove that $\forall n \geq 0,\ f(n) = (1+n)3^n$.

Base case(s):

Solution: $f(0) = 1 = (1+0)3^0$ and $f(1) = 6 = (1+1)3^1$. We need to check two base cases because the inductive step will reach back two integers.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $f(n) = (1+n)3^n$ for $n = 0, 1, \ldots, k$, for some $k \geq 1$.

Rest of the inductive step:

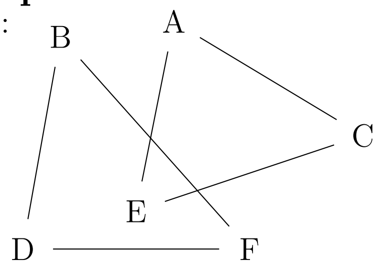
Solution: $f(k+1) = 6f(k) - 9f(k-1)$ by the definition of f. By the inductive hypothesis, we know that $f(k) = (1+k)3^k$ and $f(k-1) = k3^{k-1}$. So by substituting, we get

$$\begin{aligned}
f(k+1) &= 6(1+k)3^k - 9k3^{k-1} \\
&= 2(1+k)3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + 2 \cdot k3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + k3^{k+1} \\
&= (k+2)3^{k+1}
\end{aligned}$$

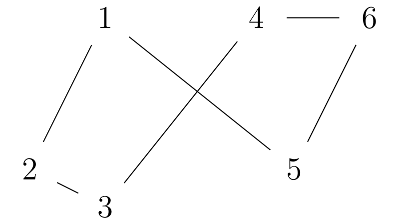
So $f(k+1) = (k+2)3^{k+1}$, which is what we needed to show.

Graphs

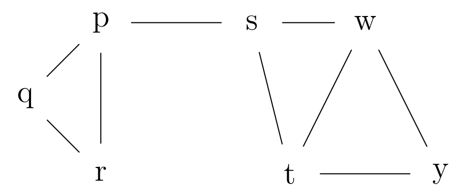
G1:



G2:



G3:



(a) How many connected components does each graph have?

Solution: G1 has two connected components. G2 and G3 each have one connected component.

(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.

Solution: No. G2 is connected, but G1 isn't connected. Also, G2 contains a cycle with 6 vertices, and G1 doesn't.

(c) What is the diameter of G3?

Solution: 4. (It's the number of edges on a shortest path between the two vertices furthest apart. In this case, y and either q or r.)

(d) Does G3 contain an Euler circuit? Why or why not?

Solution: No, it can't contain an Euler circuit because some of the vertices (e.g. p) have odd degree.

(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.

Solution: G3 contains a cut edge: the edge connecting p and s. G2 does not contain a cut edge.

(f) How many isomorphisms are there from G3 to G3? Justify your answer or show work.

Solution: p is the only degree-3 node which is connected to two degree-2 nodes. So p must map to itself. Similarly, s must map to itself because it's the only node whose neighbors all have degree 3.

However, r and q can be interchanged without changing the graph structure.

We can also interchange t and w without changing the graph structure.

So we have $2 \times 2 = 4$ isomorphisms of G3 to itself.

3. **Recursion Trees** Use a recursion tree to find the closed form expression for the function T defined by

$$T(1) = c$$
$$T(n) = 3T(n/3) + n$$

(a) At level k, how many nodes are there and what value does each contain?
Solution: Level k has 3^k nodes, each of which contains the value $n/3^k$.

(b) For input value n, what is the level of the leaf nodes?
Solution: $3^k = n$ so $k = \log_3 n$.

(c) For any non-leaf level k, what is the sum of values in the nodes?
Solution: $3^k \cdot \frac{n}{3^k} = n$.

(d) What is the total value of the leaf nodes?
Solution: cn

(e) What is the total value of all nodes, including all levels of the tree?
Solution: $\sum_{i=0}^{\log_3 n-1} n + cn = n \log_3 n + cn$

4. **Tree induction**

A Pioneer tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 5, 17, or 23.
- A node with one child contains the same number as its child.
- A node with two children contains the value $x(y+1)$, where x and y are the values in its children.

Use strong induction to prove that the value in the root of a Pioneer tree is always positive.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): **Solution:** The smallest Pioneer trees consist of a single root node, which is also a leaf. By the definition of Pioneer tree, this must contain 5, 17, or 23, all of which are positive.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that the root of a Pioneer tree of height h is always positive, for $h = 0, \ldots, k-1$.

Inductive Step:

Solution: Let T be a Pioneer tree of height k. There are two cases for what the top of T looks like.

Case 1: T consists of a root r with a single subtree S under it. r contains the same number as the root of S. Since S must be shorter than k, its root contains a positive number by the inductive hypothesis. Since r has the same label, r contains a positive number.

Case 2: T consists of a root r with a two subtrees S_1 and S_2. Suppose that the roots of S_1 and S_2 contain the numbers x and y. Then, by the definition of Pioneer tree, r contains $x(y+1)$.

Since S_1 and S_2 are shorter than k, x and y must be positive by the inductive hypothesis. Since y is positive, so is $y+1$. Since x and $y+1$ are positive, so is $x(y+1)$. So the root of T contains a positive number.

So, in either case, the root of T contains a positive number.