CS 173, Spring 2014, A Lecture, Midterm 1 Solutions

Problem 1: Multiple choice (15 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$\neg p \rightarrow q \equiv$

- $q \wedge \neg p$ ☐
- $p \vee q$ ☑ (√)
- $q \vee \neg p$ ☐
- $\neg q \wedge p$ ☐

If x is even, then $x \equiv 1 \pmod 2$

- always ☐
- sometimes ☐
- never ☑ (√)

$\forall x \in \mathbb{R},\ |x| < 0 \ \rightarrow\ x < 0$

- true ☑ (√)
- undefined truth value ☐
- false ☐

$-3 \equiv 13 \pmod 8$

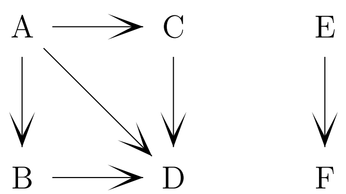
- true ☑ (√)
- false ☐

$\forall x, y \in \mathbb{Z}^+,\ lcm(y, x^2y) =$

- y ☐
- x^2y ☑ (√)
- $y \gcd(x^2, y)$ ☐
- x^2y^2 ☐

Problem 2: Short answer (16 points)

(a) (10 points) Check all boxes that correctly characterize this relation on the set $\{A, B, C, D, E, F\}$



Reflexive:	☐	Irreflexive:	$\surd$
Symmetric:	☐	Antisymmetric:	$\surd$
Transitive:	$\surd$		

(b) (6 points) Consider the function $g : \mathbb{Z}^2 \to \mathbb{R}$, $f(x, y) = x + y$. Is g onto, one-to-one, both, or neither? Briefly explain why.

Solution: g neither onto nor one-to-one.

It's not onto because g produces only integer outputs but the co-domain is all of the real numbers.

It's not one-to-one because $f(0, 1) = f(1, 0)$. [You could use any of a wide range of examples here.]

Problem 3: Short answer (12 points)

(a) (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (5y, x)$ and $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (x, x^2)$. Write the expression for $(f \circ g)(x)$, and compute $(f \circ g)(3)$. Show your work.

Solution: $(f \circ g)(x) = f(g(x)) = f(x, x^2) = (5x^2, x)$. So $(f \circ g)(3) = (5 \cdot 9, 3) = (45, 3)$.

(b) (6 points) Suppose we have the following sets:

$$\begin{aligned} A &= \{a, b, c, d, e, ..., x, y, z\} = \{\text{all 26 lowercase letters}\} \\ B &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ C &= \{10, 13\} \\ D &= \{13, 14, 15\} \end{aligned}$$

$|A \times (B \cap D)| = |A \times \emptyset| = |\emptyset| = 0$

$C \times D = \{(10, 13), (10, 13), (10, 15), (13, 13), (13, 14), (13, 15)\}$

$C \cap D = \{13\}$

Problem 4: Short answer (14 points)

(a) (6 points) Let a and b be integers, $b > 0$. We used two formulas to define the quotient q and the remainder r of a divided by b. State these two formulas.

Solution: $a = qb + r$ and $0 \leq r < b$

(b) (8 points) In $\mathbb{Z}_9$, find the value of $[5]^{21}$. You must show your work, keeping all numbers in your calculations small. **You may not use a calculator.** You must express your final answer as $[n]$, where $0 \leq n \leq 8$.

Solution: First compute powers of [5] by repeated squaring:

$[5]^2 = [25] = [7]$ $\quad$ $[5]^4 = [7]^2 = [49] = 4$ $\quad$ $[5]^8 = [4]^2 = [16] = 7$ $\quad$ $[5]^{16} = [7]^2 = [49] = 4$

Then we can compute $[5]^{21} = [5]^{16} \cdot [5]^4 \cdot [5] = ([4][4])[5] = [16][5] = [7][5] = [35] = [8]$

Problem 5: Relations and Logic (12 points)

(a) (6 points) Suppose you have a relation R on set A.

a) What do you need to show to prove that R is not anti-symmetric?

Solution: You need to find x and y in A such that $x \neq y$ and xRy and yRx

b) To show that R is an equivalence relation, which of the following properties need to be proven? (Circle those that apply.)

[reflexive] irreflexive [symmetric] anti-symmetric [transitive]

(b) (6 points) Suppose that $Cats$ is the set of all cats. Let $S(x)$ mean that cat x has stripes, $L(x)$ that a cat has long fur, and $C(x)$ that a cat is cute. Write a logical expression for the **negation** of the following statement. Your final answer should use only shorthand notation (e.g. $\forall$ and $L(x)$) and it may **not** use the $\rightarrow$ (implies) operator.

Some cats are cute but have neither long hair nor stripes.

Solution: The negation is $\neg\exists x \in Cats, C(x) \wedge \neg L(x) \wedge \neg S(x)$

That is $\forall x \in Cats, \neg(C(x) \wedge \neg L(x) \wedge \neg S(x)$

That is $\forall x \in Cats, \neg C(x) \vee L(x) \vee S(x)$

Problem 6: Proof (15 points)

Let the set A be $\{(a,b) : (a,b) \in \mathbb{R}^2, a = 3 - b^2\}$

Let the set B be $\{(x,y) : (x,y) \in \mathbb{R}^2, |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Let $(a,b) \in A$ Then, by the definition of A, a and b are real numbers and $a = 3 - b^2$. There are two cases:

Case 1: $|b| \geq 1$. Then $(a,b) \in B$ by the definition of B.

Case 2: $|b| < 1$. Then $b^2 < 1$. So then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$ and therefore $(a,b) \in B$ by the definition of B.

In either case, $(a,b) \in B$.

Since every element of A is an element of B, then $A \subseteq B$.

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Problem 7: Proof (16 points)

Suppose that n is some positive integer. Let's define the relation R_n on the integers such that aR_nb if and only if $a \equiv b + 1 \pmod{n}$. Prove the following claim

> Claim: For any integers x, y, and z, if xR_ny and yR_nz and xR_nz, then $n = 1$.

You must work directly from the definition of congruence mod k, using the following version of the definition: $x \equiv y \pmod{k}$ iff $x - y = mk$ for some integer m. You may use the following fact about divisibility: for any non-zero integers p and q, if $p \mid q$, then $|p| \leq |q|$.

Solution: Let n be a positive integer and suppose that R is as defined above. Also x, y, and z be integers and suppose that xR_ny and yR_nz and xR_nz.

By the definition of R, this means that $x \equiv y + 1 \pmod{n}$, $y \equiv z + 1 \pmod{n}$, and $x \equiv z + 1 \pmod{n}$.

Then $x - (y + 1) = kn$ $y - (z + 1) = jn$, and $x - (z + 1) = pn$, for some integers k, j, and p.

So then $x = y + 1 + kn$, $y = z + 1 + jn$ and $x = z + 1 + pn$. So $x = z + 2 + kn + jn$. So $z + 1 + pn = z + 2 + kn + jn$. So $pn = 1 + kn + jn$. So $(p - k - j)n = 1$.

We know that $p - k - j$ is an integer, so $(p - k - j)n = 1$ implies that $n|1$. Therefore $|n| \leq 1$. But n is known to be a positive integer. So n must equal 1.