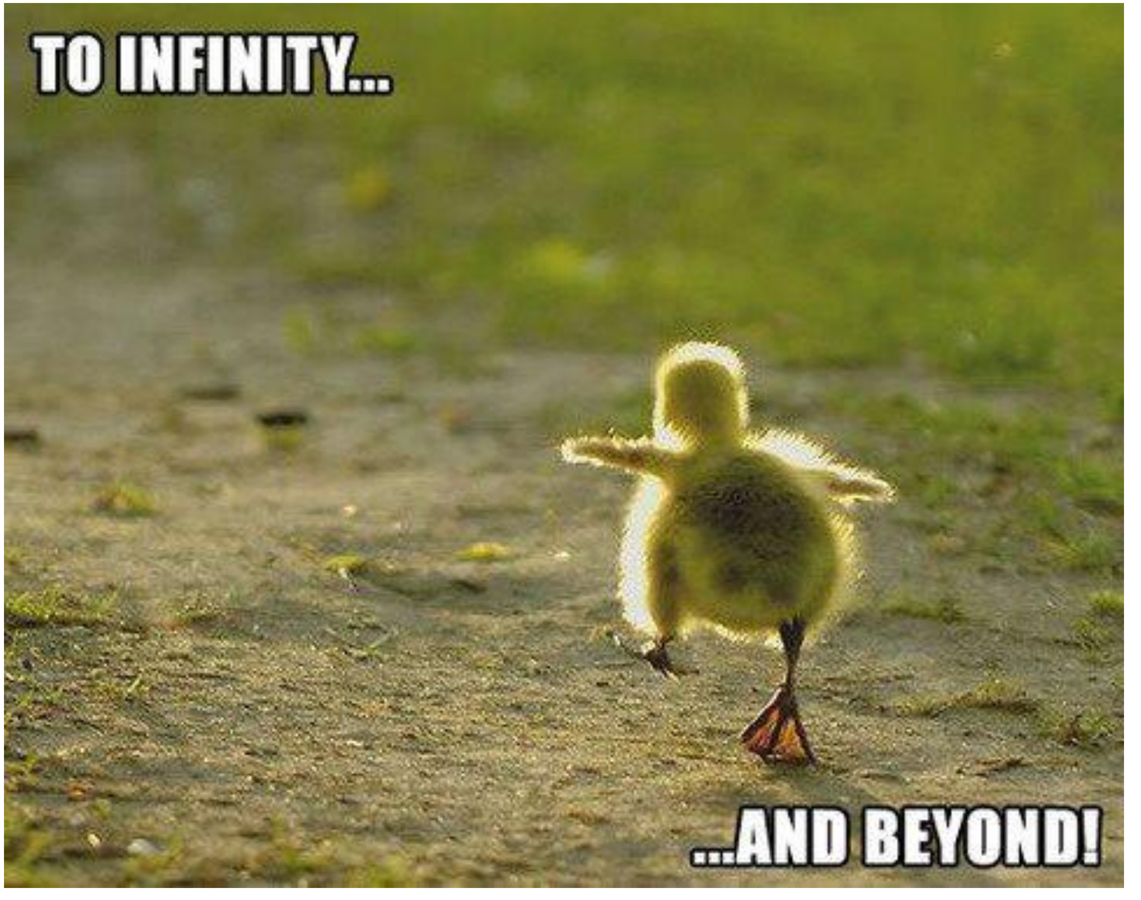
Countability



Discrete Structures (CS 173)

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Final exam times/rooms

Tuesday, **May 7, 7-10pm**

DCL 1320: Students with last names **Afridi to Mehta**

Siebel 1404: Students with last names **Melvin to Zmick**

Tell me about any conflicts asap. Note there is no specific conflict exam --- most conflicts should be resolved by other classes unless due to 3 tests in one day.

http://admin.illinois.edu/policy/code/article3_part2_3-201.html

Today's class: countability

- Are some infinite sets bigger than other infinite sets?

- How big are these common infinite sets?
  - Naturals, integers, reals, rationals, powerset of naturals

- What does it mean for a set to be "countable"?

- How do we prove that a set is or is not countable?

Are there more integers than natural numbers?

Are there more integers than natural numbers?

$|A| = |B|$ iff there is a bijection from A to B.

Are there more rational numbers than integers?

$|A| = |B|$ iff there is a bijection from A to B.

Are there more rational numbers than integers?

Cantor Schroeder Bernstein Theorem: $|A| = |B|$ iff there exist one-to-one functions $f: A \to B$ and $g: B \to A$

Countability

A set is **countably infinite** if it has the same size as the set of natural numbers.

A set is **countable** if it is finite or countably infinite.

Countable sets:

- Any subset of a countable set is countable
- The Cartesian product of finitely many countable sets is countable
- A union of countably many countable sets is countable

What sets are not countable?

Power set of natural numbers is not countable

Proof by contradiction and "diagonalization" (Cantor)

- Write a subset of natural numbers as an infinite-length binary vector
- Suppose there is a complete list of such vectors (could be infinitely many of them)
- Can construct a new vector that is different from all infinity of them!

Reals are not countable

Similar proof to for powerset of naturals

- Restrict ourselves to a subset of reals: those between 0 and 1
- Each real number is a decimal followed by a potentially infinite number of 0 to 9 numbers
- Can construct a new number by diagonalization

Which is bigger: set of real numbers or power set of natural numbers?

Diagonalization

Why doesn't the diagonalization technique work for disproving that all pairs of natural numbers is countable?

Hilbert's Paradox of the Grand Hotel

Suppose the Grand Hotel has a countably infinite number of rooms, which are all occupied.

- How can the hotel accommodate one more person without making anyone leave?

- How can the hotel accommodate a countably infinite number of new people?

The Continuum Hypothesis (CH)

Is there any set that is larger than the set of natural numbers but smaller than the set of real numbers?

- In 1931, Gödel showed that there are true statements that can't be proven true and, later, that the negation of the CH is one of them
- Thus, CH is either true or it's false but can't be proven false
- Later, Paul Cohen proved that the continuum hypothesis cannot be proved
- Thus, no logical conflict can occur from asserting the CH or its negation

Details: http://www.people.vcu.edu/~rhammack/BookOfProof/Cardinality.pdf

Summary: compare set sizes

Integer vs. Natural

Natural vs. Real

Powers of 4 vs. Integers

Real vs. Rational

Irrational vs. Rational

Powerset(Natural) vs. Real

Powerset(Real) vs. Real

Uncomputability

- A computer program is just a string (finite series of characters), so it is countable
- But the set of functions is uncountable (e.g., functions that map reals to reals)
- So there are more functions than programs – some functions cannot be computed by any program
- Implies halting problem, a topic for next class

Things to remember

- Some infinite sets are bigger than others
- We can compare sizes of infinite sets using bijections or one-to-one functions in each direction
- A “countable” set is the same size (or smaller) than the natural numbers

Next class

- Halting problem
- Conway’s game of life
- Aperiodic tilings