Functions: One-to-One and Permutations

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Last class: functions

- Function: maps each input element to exactly one output element and every function has
  - a **type signature** that defines what inputs and outputs are possible
  - an **assignment** or mapping that specifies which output goes with each input

$$f : A \to B, f(x) = \dots$$

- Two functions are **equal** iff both the type signature and the assignment are be the same
- A function is **onto** iff every output element is assigned at least once.

Nested quantifiers

- There is a pencil for every student.

$$\forall s, \exists p, sHp$$

- Every student is using the same pencil.

$$\exists p, \forall p, sHp$$

- For a function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$, every output element is assigned to at least one input element.

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = x^2 = y$$

- For a function $f : A \to B$, there is one output that is assigned to every input.

$$\exists y \in B, \forall x \in A, f(x) = y \quad f : \mathbb{R} \to \mathbb{R}, f(x) = 0$$

Nested quantifiers

- There exists an $x \in \mathbb{R}$, such that for every $y \in \mathbb{R}, y = x^2$ False
- There exists a $y \in \mathbb{R}$, such that for every $x \in \mathbb{R}, y = x^2$ False
- For every $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$, such that $y = x^2$ True
- For every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $y = x^2$ False

Negation with nested quantifiers

It's not true that there is a pencil for every student.

$$\neg\forall s, \exists p, sHp \equiv \exists s, \neg\exists p, sHp \equiv \exists s, \forall p, s \not{H} p$$

No student has any pencil.

$$\forall s, \forall p, s \not{H} p \equiv \neg\exists s, \exists p, sHp$$

There is one pencil that no student has.

$$\forall s, \exists p, s \not{H} p$$

onto: $f : A \to B, f(x)$ is onto iff $\forall y \in B, \exists x \in A,$ s.t. $f(x) = y$.
not onto: $\neg\forall y \in B, \exists x \in A, f(x) = y \equiv$
$\exists y \in B, \neg\exists x \in A, f(x) = y \equiv \exists y \in B, \forall x \in A, f(x) \neq y$

Disproof of onto

Disprove: $f : \mathbb{N} \to \mathbb{N}, f(x) = x + 2$ is onto.
Definition: $f : A \to B, f(x)$ is onto iff $\forall y \in B, \exists x \in A$, s.t. $f(x) = y$.

Negation of onto: $\exists y \in B, \forall x \in A, f(x) \neq y$
We need to show there exists a $yin\mathbb{N}$ s.t. there is no $x \in \mathbb{N}$ with $f(x) = x + 2 = y$.
Suppose $y = 0$. $y \in \mathbb{N}$. Now $x = y - 2 = -2$. but $-2 \notin \mathbb{N}$ so there is no x s.t. $f(x) = 0$ and f is not onto. QED

Today

- When is a function "one-to-one" or "bijective"?
- What is the inverse of a function?
- The Pigeonhole Principle
- Permutations and their applications

One-to-one

x is a **preimage** of y if $f(x) = y$.

One-to-one: no two inputs map to the same output (no output has more than one preimage)

$$f : A \to B, \forall x, y \in A, (x \neq y) \to (f(x) \neq f(y))$$

contrapositive?

$$f : A \to B, \forall x, y \in A, (f(x) = (f(y)) \leftrightarrow (x = y)$$

Proof of one-to-one

Claim: $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x - 1$ is one-to-one.
Definition: $f : A \rightarrow B, f(x)$ is one-to-one if and only if
$\forall x, y \in A, (f(x) = f(y)) \rightarrow (x = y)$

Let $x, y \in \mathbb{R}$ st $f(x) = f(y)$.

$$2x - 1 = 2y - 1$$

$$2x = 2y$$

$$x = y$$

Thus f is one-to-one by definition. QED

Proof of one-to-one

Claim: For any sets A, B, C and functions $f : A \to B, g : B \to C$, if f and g are one-to-one, then $g \circ f$ is also one-to-one.
Definition: $f : A \to B, f(x)$ is one-to-one if and only if $\forall x, y \in A, (f(x) = f(y)) \to (x = y)$

Suppose there are sets A, B, C and functions $f : A \to B, g : B \to C$, st f and g are one-to-one.
Suppose $x, y \in A$ st $g(f(x)) = g(f(y))$.
Since g is one-to-one

$$f(x) = f(y)$$

Since f is one-to-one

$$x = y$$

There for by definition $g \circ f$ is one-to-one.

A function is **bijective** if it is onto and one-to-one.

- Inverse function
  if $f : A \to B, f(x) = y$, then $f^{-1} : B \to A, f^{-1}(y) = x$.

$$\forall x \in A, f^{-1}(f(x)) = x$$

Pigeonhole principle

- Pigeonhole principle: if you have more objects than labels, then at least two objects must get the same label
- In the spring this class had 400 students. Was every day someone's birthday? Do two students have the same birthday?
- Claim: The first 50 powers of 13 include at least two numbers whose difference is a multiple of 47.
- Claim Suppose n people are at a party and everyone has at least one admirer. At least two people will have the same number of admirers.

Proof with bijective, pigeonhole

Claim: If $f : A \to B$ is bijective, then $|A| = |B|$
Definition: $f : A \to B$, is one-to-one iff every output is assigned at most once.
Definition: $f : A \to B, f(x)$, is onto iff every output is assigned at least once

Contrapositive: $|A| \neq |B|$ then $f : A \to B$ is not bijective.
Proof by cases:
Case $|A| < |B|$: If f has fewer input elements than the output elements, some elements must not be covered so f is not onto.
Case $|A| > |B|$: If f has more input elements then output elements, some output value must be assigned by more then one input element therefore f is not one to one.
Since all cases f is not one-to-one or onto so we have proven the claim. QED

Ordered selection

- Suppose I have 6 gems, and you get to choose 1. How many different combinations of gems can you choose?

$$6$$

- Suppose I have n gems and want to put them in a row from left to right. How many different ways can I arrange them?

$$n!$$

- Suppose I have 6 gems and want to put three of them in a row from left to right. How many different ways can I arrange them?

$$6 \times 5 \times 4$$

Permutations

Suppose $f : A \to B$ with $|A| = k$ and $|B| = n$. How many different one-to-one functions can I create? $P(n, k)$

$$\frac{n!}{(n-k)!}$$

How many ways can I rearrange the letters in "nan"?

$$\frac{3!}{2!}$$

How many ways can I rearrange the letters in "yellowbelly"?

$$\frac{11!}{4!2!2!}$$

Things to remember

- One-to-one: no two inputs are assigned to the same output
- Pigeonhole principle: if you have more objects than labels, some objects must get the same label
- Permutations: number of ordered combinations $\frac{n!}{(n-k)!}$ divided by number of equivalent orderings
- Midterm 1 on Monday review on Thursday.

Proof with one-to-one

Claim: Strictly increasing functions are one-to-one.
Definition: $f : A \to B, f(x)$ is one-to-one if and only if
$\forall x, y \in A, (f(x) = f(y)) \to (x = y)$
Definition: $f : A \to B, f(x)$ is strictly increasing iff
$\forall x, y \in A, (x < y) \to (f(x) < f(y))$

Contrapositive: $x \neq y \to f(x) \neq f(y)$
Let $x, y \in A$ s.t. $x \neq y$
WLOG $x < y$, Since f is strictly increasing $f(x) < f(y)$ and thus
$f(x) \neq f(y)$ which proves the claim. QED