Induction Tutorial Solutions

11.1 Simple examples

b) We proceed by induction on n.

Base: Let $n = 1$. Then $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{(1+1)}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for each $n \in \mathbb{Z}^+$ less than some positive integer r. Then our goal is to show $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$.

$$\begin{aligned}
\sum_{k=1}^{r} \frac{1}{k(k+1)} &= \sum_{k=1}^{r-1} \frac{1}{k(k+1)} + \frac{1}{r(r+1)} && \text{(pulling a term out of the summation)} \\
&= \frac{(r-1)}{(r-1)+1} + \frac{1}{r(r+1)} && \text{(by the Inductive Hypothesis)} \\
&= \frac{(r-1)}{r} + \frac{1}{r(r+1)} && \text{(this and remaining steps are just algebra)} \\
&= \frac{(r-1)(r+1)+1}{r(r+1)} \\
&= \frac{r^2}{r(r+1)} \\
&= \frac{r}{r+1}
\end{aligned}$$

Thus (by transitivity of equality) $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$, QED.

c) *The IH on top of the equals sign below is a shorthand for showing where you are applying the Inductive Hypothesis.*

Proof by induction on n.

Base: Let $n = 0$. Then $(\sum_{i=0}^{0} i)^2 = 0 = \sum_{i=0}^{0} i^3$. ✓

Induction: Fix k and suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \cdots, k-1$. Then we

get the following:

$$\begin{aligned}
(\sum_{i=0}^{k} i)^2 &= (\sum_{i=0}^{k-1} i + k)^2 \\
&= (\sum_{i=0}^{k-1} i)^2 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&\overset{IH}{=} \sum_{i=0}^{k-1} i^3 + 2k(\sum_{i=0}^{k-1} i) + k^2 \\
&= \sum_{i=0}^{k-1} i^3 + 2k\frac{(k-1)k}{2} + k^2 && \text{(by the given hint)} \\
&= \sum_{i=0}^{k-1} i^3 + k^3 \\
&= \sum_{i=0}^{k} i^3
\end{aligned}$$

Induction complete.

11.2 Induction with congruences

Fix $a, b \in \mathbb{Z}$ and $p \in \mathbb{Z}^+$. Now we need to show $\forall n \in \mathbb{Z}^+, P(n)$, where $P(n)$ is "if $a \equiv b \pmod{p}$ then $a^n \equiv b^n \pmod{p}$". We proceed by induction on n:

Base: We need to show $P(1)$, i.e. that if $a \equiv b \pmod{p}$ then $a^1 \equiv b^1 \pmod{p}$. But this is clearly true since $a = a^1$ and $b = b^1$.

Induction: Fix z, and suppose (as our Inductive Hypothesis) that for any i with $1 \leq i < z$, $P(i)$ is true. Now we need to show $P(z)$ is true, i.e. we need to show that if $a \equiv b \pmod{p}$ then $a^z \equiv b^z \pmod{p}$.

So suppose (towards direct proof) that $a \equiv b \pmod{p}$. Using this fact along with $P(z-1)$ (which is true by the IH), we also know that $a^{z-1} \equiv b^{z-1} \pmod{p}$. Multiplying our two equivalences together gives us $a \cdot a^{z-1} \equiv b \cdot b^{z-1} \pmod{p}$. This in turn gives us $a^z \equiv b^z \pmod{p}$, QED.

(Commentary: notice that the original claim has four variables in it - a, b, n, p. It would be valid to attempt an induction proof using any of those four as the induction variable, but if you pick something other than n in this case you will discover that there is no good way to finish the proof. So as always, don't be afraid to switch tactics if your current path seems not to be working.)

11.4 A broken induction proof

(Commentary: Obviously the proof must be wrong since the claim it is proving is clearly false. While that is not enough to say where *the flaw in the proof is, it does give us a good*

place to check: $P(1)$ is true but $P(2)$ is false, so we should look at the inductive step and carefully audit its argument that $P(1) \to P(2)$.)

The argument implicitly relies on the fact that S' and S'' are not disjoint. If the sets overlap by even one horse H_*, then the proof is correct that all horses in the union are the same color, since all the horses in S' are H_*'s color and so are the horses in S''. However, consider the argument in the inductive step when $k = 2$. In this case, $S' = \{H_2\}$ and $S'' = \{H_1\}$, which are disjoint. Thus while it is true that all the horses in S' are the same color and all the horses in S'' are the same color, it is wrong for the proof to claim from this that all the horses in the union must also be the same color.

Induction-Like Implications

Since we're going to need every single P we come up with to satisfy $\forall k, P(k) \to P(k+2)$, let's first look at a few examples of such predicates. Remember that the only time $P(k) \to P(k+2)$ is false is if $P(k)$ is true AND $P(k+2)$ is false. We thus have a lot of leeway when designing P, including useful simple extremes where P is always false or always true. Also, for any z, $P(n) = n \geq z$ will work, because then whenever $P(k)$ is true, $k+2$ is bigger than k so $P(k+2)$ is also true. Other predicates that work include "n is even", "n is odd", and a conjunction or disjunction ("and" and "or") of any two other predicates that work.

I'll just give one answer for each prompt but there are usually many other options.

c) S is true for $P(n) :=$ "False", and false for $P(n) :=$ "True". *(read ":=" as "is defined to be")*

d) S is false for *any* P that satisfies the $P(k) \to P(k+2)$ rule: from $(\forall n \leq 100(P(n))$ we know $P(100)$ would have to be true, but from $(\forall n > 100(\neg P(n))$ we know $P(102)$ would have to be false.

e) S is true for $P(n) :=$ "$n \geq 101$", and false for $P(n) :=$ "False".

f) S is true for $P(n) :=$ "True", and false for $P(n) :=$ "n is even". Note that $\forall n(P(n+2))$ is equivalent (in our given universe of nonnegative integers) to $\forall n \geq 2, (P(n))$

h) S is true for *any* P which satisfies the $P(k) \to P(k+2)$ rule), since we have one of two cases:

- $P(1)$ is false. In this case the whole statement is vacuously true.
- $P(1)$ is true. In this case, $P(3)$ must also be true (since $P(1) \to P(1+2)$), then $P(5)$ must also be true (since $P(3) \to P(3+2)$), etc for all the odd natural numbers, so we get that P is true for all odd naturals, i.e. for any number that can be written as $2n+1$ for some $n \in \mathbb{N}$. *(Formalizing this argument would require induction, though it would be a bit awkward to formalize since our goal would be to induct over only the odd numbers.)*

The Diagonal Robot

Let $P(n)$ be the claim "After the robot takes n steps, it is guaranteed to be at a location (x, y) with $x + y$ even." If we can prove that $P(n)$ is true for all $n \in \mathbb{N}$, then we are done, because $(0, 1)$ has an odd sum of coordinates. So we proceed by induction on n:

Base: After the robot takes zero steps, it is still at its starting location of $(1, 1)$, and $1+1$ is even.

Induction: Suppose $P(n)$ is true for $n = 0, 1, \cdots, k$. Then we need to show $P(k + 1)$, i.e. that after the robot takes $k + 1$ steps, it is still guaranteed to be at a location with even coordinate sum. So consider an arbitrary case where the robot has taken $k + 1$ steps. One step earlier, it had taken k steps, and by the inductive hypothesis, it was at a location (x, y) with $x + y$ even. So now after the $(k + 1)$st step, the only four places it can be are $(x + 1, y + 1)$, $(x + 1, y - 1)$, $(x - 1, y + 1)$, and $(x - 1, y - 1)$. In each of those cases the sum is either $x + y + 2$, $x + y$, or $x + y - 2$, so since $x + y$ is even, the new sum is also even.