Week 14 Tutorial Solutions

19.1 Which Kind of Infinity?

A common fast way to show that a set is countable is to note that every element in the set has a finite representation. Be careful trying to use this to show that a set is uncountable because even if the representations aren’t finite, there may be alternate representations that are.

a) **Countably infinite.** In fact it’s basically the definition of countably infinite - the bijection mapping it to \( \mathbb{N} \) is \( \text{id}_\mathbb{N} \).

b) **Uncountable.** The powerset of a set always has a (strictly) larger cardinality than that set. *(Thinking with representations: these do not appear to all have finite representations - if I have an infinite set of naturals with no pattern, how would I possibly write down that set?)*

c) **Uncountable.** We know \( \mathbb{R} \) is uncountable, and \( \mathbb{R} \subseteq \mathbb{C} \).

d) **Countably infinite.** We can provide a one-to-one function \( f \) mapping these to the (finite) bit strings: given \( S \) with maximum element \( n \), return the bit string of length \( n + 1 \) with a 1 in (0-indexed) position \( i \) iff \( i \in S \). For example, \( f(\{0, 3, 4\}) = 10011 \). And we know the set bit strings (or any other strings with a finite alphabet) are countable. *(Thinking with representations: each \( S \in X \) has a roster notation which is finite - e.g. \{0,3,4\}.)*

e) **Countably infinite.** Each book is just one finite string using a finite alphabet. *(You may be tempted to think of a book as a list of strings separated by spaces, but that’s making it more complicated than necessary.)*

f) **Countably infinite.** We know \( \mathbb{Q} \) is countable, and this set is a subset of \( \mathbb{Q} \). *(Thinking with representations: these are reals specifically chosen to have expansions that end - i.e. representations that are finite.)*

19.2 A Curious Bijection

a)
b) Consider the values of $x, y$ satisfying $x + y = k$.

Because we are in $\mathbb{N}$, for any such values of $x$ and $y$ we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.

Thus, $x$ ranges from 0 to $k$, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k + 1)}{2}$, we can also write this as:

$$\frac{k(k + 1)}{2} \leq f(x, y) \leq \frac{k(k + 1)}{2} + k$$

c) The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.

We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all $x, y$, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of $x$ and $y$ shows that $(2, 3)$ is the only element in the pre-image of 17.

It would also have been sufficient to prove that $f$ is one-to-one, but this takes considerably more effort.

d) Let $k = x + y$, $l = p + q$. We are assuming that $k \neq l$. So, without loss of generality, assume that $k < l$. (If $k$ was bigger than $l$, we could just swap the names of the two variables.) We aim to show that $f(x, y) < f(p, q)$.

From the solution to b), we know that the sums of the coordinates $k$ and $l$ restrict the output values to very limited ranges. So, $f(x, y)$ has to be no bigger than the upper end of the range of outputs for the sum $k = x + y$. That is:

$$f(x, y) \leq \frac{k(k + 1)}{2} + k$$

Similarly, $f(p, q)$ has to be at least as big as the lower end of the range of outputs for the sum $l = p + q$. That is:
Thus, to show that \( f(x, y) < f(p, q) \), it suffices to show that \( \frac{k(k+1)}{2} + k < \frac{l(l+1)}{2} \). Since \( k < l \), we have that \( k + 1 \leq l \) and therefore, substituting \( k + 1 \) for \( l \), we have that \( \frac{(k+1)(k+2)}{2} \leq \frac{l(l+1)}{2} \). It therefore suffices to show that \( \frac{k(k+1)}{2} + k < \frac{(k+1)(k+2)}{2} \), which we do as follows:

\[
\frac{k(k+1)}{2} + k = \frac{k(k+1)+2k}{2} = \frac{k^2+3k}{2} < \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}
\]

e) Assume the contrary, that \( f(x, y) = f(p, q) \). Further, let \( k = x + y = p + q \). Then:

\[
\begin{align*}
f(x, y) &= f(p, q) \\
s(x + y) + x &= s(p + q) + p \\
s(k) + x &= s(k) + p \\
x &= p
\end{align*}
\]

Since \( x = p \) and \( x + y = p + q \), we have that \( y = q \). But we assumed that \( (x, y) \neq (p, q) \), contradiction.

**Additional problem**

Lemma: For (non-empty) sets \( A \) and \( B \), there exists a one-to-one function \( f : A \to B \) if and only if there exists an onto function \( g : B \to A \).

Proof: See solution to week 5's additional tutorial problem - the only difference is that now we are working with arbitrary sets instead of subsets of \( \mathbb{N} \), so where that solution uses the function \( \text{minimum} \) (which can choose a representative from a set of naturals), we instead have to use the choice function \( h \) from the hint. \( \square \)

We know that by definition, there exists a one-to-one function \( f : A \to B \) if and only if \( |A| \leq |B| \). So now by the lemma, we've established that there exists an onto function \( g : B \to A \) if and only if \( |A| \leq |B| \).