1. (8 points) Consider the following grammar $G$, with start symbol $S$ and terminals $a$ and $b$.

$$S \rightarrow a\,S\,a \mid b\,S\,b \mid a\,S\,b \mid b\,S\,a \mid a \mid b$$

Amy claims that this generates all non-empty strings containing a’s and/or b’s. Is this correct? Justify your answer.

**Solution:**

Amy is wrong. This grammar only generates strings of odd length.

2. (4 points) Check the (single) box that best characterizes each item.

Total number of leaves in a full and complete 5-ary tree of height $h$

- $5^h$ [ ]
- $\leq 5^h$ [ ]
- $\geq 5^h$ [ ]
- $5^{h+1} - 1$ [ ]

The level of a leaf node in a full and complete binary tree of height $h$.

- $0$ [ ]
- $1$ [ ]
- $h - 1$ [ ]
- $\leq h$ [ ]
- $h$ [ ]
1. (8 points) Here is a grammar with start symbol $S$ and terminal symbols $a$ and $b$. Draw three parse trees for the string $abba$ that match this grammar.

$$S \rightarrow SS \mid aS \mid Sa \mid b$$

Solution:

2. (4 points) Check the (single) box that best characterizes each item.

A full $m$-ary tree with $i$ internal nodes has $mi + 1$ nodes total.  
always ✓  sometimes  never

A binary tree of height $h$ has at least $2^{h+1} - 1$ nodes.  
true  false ✓
1. (8 points) Consider the following grammar $G$

$$S \rightarrow S \ b \ S \ | \ a \ | \ c \ d$$

$S$ is the only start symbol. The terminal symbols are $a$, $b$, $c$, and $d$.

Here are two sequences of leaf labels. For each sequence, either draw a tree from grammar $G$ whose leaves have this sequence of labels, or else explain briefly why $G$ cannot generate this sequence of leaf labels.

**aaacd**

**Solution:** In grammar $G$, making strings with more than two leaves requires using the first rule ($SbS$) which produces a $b$. This string can’t be generated by $G$ because it is more than two characters long with no $b$ in it.

**bbbbb**

**Solution:** Impossible. Since the only terminal in the string is $b$, the only rule we could be using is $S \rightarrow SbS$. But each time we use this rule, the count of $S$ nodes without children increases by one. This is a problem, since $S$ nodes can’t be leaves.

2. (4 points) Check the (single) box that best characterizes each item.

The mathematical symbol for an empty (zero-length) string

- $\emptyset$  
- $e$  
- $\epsilon$  
- NULL

Number of bit strings of length $\leq k$.

- $2^k$  
- $2^k - 1$  
- $2^{k-1}$  
- $2^{k+1} - 1$
1. (8 points) Min’s virus detection code needs to generate all strings of the form $a^n b^n$. That is, all strings that consist of a sequence of one or more a’s followed by the same number of b’s. Write a context-free grammar $G$ that will do this.

Solution:

$G$ has start symbol $S$, terminals $a$ and $b$, and the following rules:

$$S \rightarrow a \ S \ b \mid a \ b$$

2. (4 points) Check the (single) box that best characterizes each item.

The number of nodes in a binary tree of height $h$

- $\geq 2^h$ [ ]
- $2^h+1 - 1$ [ ]
- $\leq 2^{h+1} - 1$ [ ]
- $\geq 2^{h+1} - 1$ [ ]

A tree node is a descendent of itself.

- always [ ]
- sometimes [ ]
- never [ ]
1. (8 points) Consider the following grammar $G$

   $S \rightarrow a\ S\ b \ | \ b\ S\ b \ | \ c$

   $S$ is the only start symbol. The terminal symbols are $a$, $b$, and $c$.

   Here are two sequences of leaf labels. For each sequence, either draw a tree from grammar $G$ whose leaves have this sequence of labels, or else explain briefly why $G$ cannot generate this sequence of leaf labels.

   **ababb**

   **Solution:**

   This is impossible. In strings produced by $G$, the middle character must be a $c$.

   **babcbbb**

   **Solution:**

   

   

   

   

   

   

   

   

2. (4 points) Check the (single) box that best characterizes each item.

   The level of the root node in a tree of height $h$.

   

   

   

   

   

   

   

   A tree node is a proper ancestor of itself.

   always □ sometimes □ never □
1. (8 points) Here is a grammar with start symbol $S$ and terminal symbols $a$, $b$, and $c$. Circle the trees that match the grammar.

$$S \rightarrow a \ N \ a \mid b \ N \ b \mid a \mid b$$

$$N \rightarrow S \ S \mid c$$

2. (4 points) Check the (single) box that best characterizes each item.

A binary tree of height $h$ has at least $2^h - 1$ nodes.  
true  [ ]  false  [√]

A full $m$-ary tree with $i$ internal nodes has $mi - 1$ nodes total.  
$mi - 1$  [ ]  $mi$  [ ]  $mi + 1$  [√]  $\leq mi + 1$  [ ]
1. (8 points) Consider the following grammar $G$

$$
S \rightarrow a \ S \ b \ | \ b \ S \ b \ | \ a \ | \ b
$$

$S$ is the only start symbol. The terminal symbols are $a$ and $b$.

Here are two sequences of leaf labels. For each sequence, either draw a tree from grammar $G$ whose leaves have this sequence of labels, or else explain briefly why $G$ cannot generate this sequence of leaf labels.

**bababbb**

**Solution:**

- $S$
  - $b$
  - $S$
  - $b$
- $a$
  - $S$
  - $b$
- $b$
  - $S$
  - $b$
  - $a$

**aaaab**

**Solution:**

This is impossible. In a string produced by grammar $G$, all characters after the middle of the string must be $b$’s.

2. (4 points) Check the (single) box that best characterizes each item.

- The number of leaves in a binary tree of height $h$
  - $2^h$
  - $2^{h+1} - 1$
  - $\geq 2^h$
  - $\leq 2^h$
  - $\square$

- The number of paths between two distinct nodes in an $n$-node tree. Paths in opposite directions count as the same.
  - $n$
  - $2n$
  - $\frac{n(n-1)}{2}$
  - $n^2$
  - $\frac{n(n+1)}{2}$
  - $\square$
  - $\square$
  - $\square$
  - $\square$
1. (8 points) Here is a grammar with start symbol $S$ and terminal symbols $a$, $b$, $c$, and $d$. Circle the trees that match the grammar.

$$
S \rightarrow a\ N\ b \mid a\ N\ c \mid a \\
N \rightarrow S\ S \mid d
$$

2. (4 points) Check the (single) box that best characterizes each item.

The diameter of a tree of height $h$.

- $\leq h$ \[\square\] $h$ \[\square\] $h+1$ \[\square\]

- $2h$ \[\square\] $\leq 2h$ \[\checkmark\]

The number of nodes in a full complete binary tree of height $h$.

- $\geq 2^h$ \[\square\] $2^{h+1} - 1$ \[\checkmark\]

- $\leq 2^{h+1} - 1$ \[\square\] $\geq 2^{h+1} - 1$ \[\square\]