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(20 points) Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by

$g(0) = 0 \qquad g(1) = \frac{4}{3}$

$g(n) = \frac{4}{3}g(n-1) - \frac{1}{3}g(n-2), \quad \text{for } n \geq 2$

Use (strong) induction to prove that $g(n) = 2 - \dfrac{2}{3^n}$

Solution: Proof by induction on n.

Base case(s): $n = 0$: $2 - \frac{2}{3^n} = 2 - \frac{2}{1} = 0 = g(0)$ So the claim holds.

$n = 1$: $2 - \frac{2}{3^n} = 2 - \frac{2}{3} = \frac{4}{3} = g(1)$ So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $g(n) = 2 - \frac{2}{3^n}$, for $n = 0, 1, \cdots, k-1$ for some integer $k \geq 2$.

Inductive Step:

We need to show that $g(k) = 2 - \dfrac{2}{3^k}$

$$\begin{aligned}
g(k) &= \frac{4}{3}g(k-1) - \frac{1}{3}g(k-2) && \text{[by the def, } k \geq 2\text{]} \\
&= \frac{4}{3}\left(2 - \frac{2}{3^{k-1}}\right) - \frac{1}{3}\left(2 - \frac{2}{3^{k-2}}\right) && \text{[Inductive Hypothesis]} \\
&= \frac{8}{3} - \frac{8}{3^k} - \frac{2}{3} + \frac{2}{3^{k-1}} \\
&= \frac{6}{3} - \frac{8}{3^k} + \frac{6}{3^k} \\
&= 2 - \frac{2}{3^k}.
\end{aligned}$$

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(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 2$

$f(1) = 7$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$, for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 2(3 \cdot 2^{k-2} + (-1)^{k-1}) && \text{by inductive hypothesis} \\
&= (3 \cdot 2^{k-1} + (-1)^k) \quad + \quad 3 \cdot 2^{k-1} + 2(-1)^{k-1} \\
&= 6 \cdot 2^{k-1} + (-1)^k - 2(-1)^k \\
&= 3 \cdot 2^k - (-1)^k \\
&= 3 \cdot 2^k (-1)^{k+1}
\end{aligned}$$

So $f(k) = 3 \cdot 2^k (-1)^{k+1}$, which is what we needed to show.

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(20 points) Use (strong) induction to prove that the following claim holds:

Claim : For any integer $n \geq 2$, if $p_1, \ldots, p_n$ is a sequence of integers and $p_1 < p_n$, then there is an index j $(1 \leq j < n)$ such that $p_j < p_{j+1}$.

Solution:

Base case(s): Proof by induction on n. At $n = 2$: It's given that $p_1 < p_n$. But $p_n = p_2$. So $p_1 < p_2$ and so $j = 1$ is the required index.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that any sequence of integers $p_1, \ldots, p_n$ with $p_1 < p_n$ has an index j $(1 \leq j < n)$ such that $p_j < p_{j+1}$, for $n = 2, \ldots, k$.

Rest of the inductive step: Let $p_1, \ldots, p_{k+1}$ be a sequence of $k + 1$ integers, with $p_1 < p_{k+1}$.

Consider p_k and p_{k+1}. There are two cases:

Case (1): $p_k < p_{k+1}$. Then the index $j = k$ works.

Case (2): $p_k \geq p_{k+1}$. Then we have $p_1 < p_{k+1}$ and $p_{k+1} \leq p_k$. So $p_1 < p_k$. So we can apply the inductive hypothesis to the shorter subsequence $p_1, \ldots, p_k$. That is, by the inductive hypothesis, there is an index j into the subsequence (i.e. $1 \leq j < k$) such that $p_j < p_{j+1}$. This (obviously) also works as an index into the longer sequence of $k + 1$ integers.

In both cases, we have found an index j such that $p_j < p_{j+1}$, which is what we needed to find.

[Notes: it also works to remove the first element p_1 from the sequence, with small changes to the inductive step. Your inductive step doesn't need to be quite this detailed.]

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(20 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by is defined by

$f(1) = 5 \qquad f(2) = -5$

$f(n) = 4f(n-2) - 3f(n-1)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 2 \cdot (-4)^{n-1} + 3$

Solution: Proof by induction on n.

Base case(s): For $n = 1$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^0 + 3 = 2 \cdot 1 + 3 = 5$, which is equal to $f(1)$.
For $n = 2$, $2 \cdot (-4)^{n-1} + 3 = 2 \cdot (-4)^1 + 3 = 2 \cdot (-4) + 3 = -5$, which is equal to $f(2)$.
So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $f(n) = 2 \cdot (-4)^{n-1} + 3$, for $n = 1, 2, \ldots, k-1$, for some integer $k \geq 3$

Rest of the inductive step:
Using the definition of f and the inductive hypothesis, we get
$f(k) = 4f(k-2) - 3f(k-1) = 4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3)$
Simplifying the algebra,

$$
\begin{aligned}
4(2 \cdot (-4)^{k-3} + 3) - 3(2 \cdot (-4)^{k-2} + 3) &= 8 \cdot (-4)^{k-3} + 12 - 6 \cdot (-4)^{k-2} - 9 \\
&= -2 \cdot (-4)^{k-2} - 6 \cdot (-4)^{k-2} + 3 \\
&= -8 \cdot (-4)^{k-2} + 3 = 2 \cdot (-4)^{k-1} + 3
\end{aligned}
$$

So $f(k) = 2 \cdot (-4)^{k-1} + 3$, which is what we needed to prove.

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(20 points) Suppose that θ is a constant (but unknown) real number. For any real number p, the angle addition formulas imply the following two equations (which you can assume without proof):

$$\cos(\theta)\cos(p\theta) = \cos((p+1)\theta) + \sin(\theta)\sin(p\theta) \quad (1)$$
$$\cos(\theta)\cos(p\theta) = \cos((p-1)\theta) - \sin(\theta)\sin(p\theta) \quad (2)$$

Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by

$f(0) = 1$ $\qquad f(1) = \cos(\theta)$

$f(n+1) = 2\cos(\theta)f(n) - f(n-1)$, for all $n \geq 2$.

Use (strong) induction to prove that $f(n) = \cos(n\theta)$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(n) = f(0) = 1 = \cos(0) = \cos(0\theta) = \cos(n\theta)$.

At $n = 1$, $f(n) = f(1) = \cos\theta = \cos(1\theta) = \cos(n\theta)$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

$f(n) = \cos(n\theta)$ for $n = 0, \ldots, k$.

Rest of the inductive step: In particular, by the inductive hypothesis, $f(k) = \cos(k\theta)$ and $f(k-1) = \cos((k-1)\theta)$.

If we set $p = k$ in equations (1) and (2), and then add them together, we get

$$2\cos(\theta)\cos(k\theta) = \cos((k+1)\theta) + \cos((k-1)\theta)$$

So then we can compute

$$\begin{aligned} f(k+1) &= 2\cos(\theta)f(k) - f(k-1) \\ &= 2\cos(\theta)\cos(k\theta) - \cos((k-1)\theta) \qquad \text{(by the IH)} \\ &= \cos((k+1)\theta) + \cos((k-1)\theta) + \cos((k-1)\theta) \\ &= \cos((k+1)\theta) \end{aligned}$$

So $f(k+1) = \cos((k+1)\theta)$, which is what we needed to show.

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(20 points) A Zellig graph consists of $2n$ ($n \geq 1$) nodes connected so as to form a circle. Half of the nodes have label 1 and the other half have label -1. As you move clockwise around the circle, you keep a running total of node labels. E.g. if you start at a 1 node and then pass through two -1 nodes, your running total is -1. Use (strong) induction to prove that there is a choice of starting node for which the running total stays ≥ 0.

Hint: remove an adjacent pair of nodes.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, there are only two nodes. If you start at the node with label 1, the running total stays ≥ 0.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there is a choice of starting node for which the running total stays ≥ 0, for Zellig graphs with $2n$ nodes, where $n = 1, \ldots, k-1$.

Rest of the inductive step: Let G be a Zellig graph with $2k$ nodes. Find a 1 node that immediately precedes a -1 (going clockwise). Remove those two nodes m and s from G to create a smaller graph H.

By the inductive hypothesis, we can find a starting node p on H such that the running total stays ≥ 0. I claim that p also works as a starting node for G. Between p and m, we see the same sequence of nodes as in H, so the total stays ≥ 0. The total increases by 1 at m and the immediately decreases by 1 at s. So it can't dip below zero in that section of the circle. Between s and returning to p, we have the same running totals as in H.

So G has a starting point for which all the running totals stay ≥ 0, which is what we needed to prove.

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(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2 \qquad f(1) = 5 \qquad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 3$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $f(0) = 2$ and $1 - 2^n + 2 \cdot 3^n = 1 - 1 + 2 = 2$

At $n = 1$, $f(1) = 5$ and $1 - 2^n + 2 \cdot 3^n = 1 - 2 + 6 = 5$

At $n = 2$, $f(2) = 15$ and $1 - 2^n + 2 \cdot 3^n = 1 - 4 + 18 = 15$

So the claim holds at all three values.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 1 - 2^n + 2 \cdot 3^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: By the definition of f and the inductive hypothesis, we get

$$
\begin{aligned}
f(k) &= 6f(k-1) - 11f(k-2) + 6f(k-3) \\
&= 6(1 - 2^{k-1} + 2 \cdot 3^{k-1}) - 11(1 - 2^{k-2} + 2 \cdot 3^{k-2}) + 6(1 - 2^{k-3} + 2 \cdot 3^{k-3}) \\
&= (6 - 11 + 6) - (6 \cdot 2^{k-1} - 11 \cdot 2^{k-2} + 6 \cdot 2^{k-3}) + 2(6 \cdot 3^{k-1} - 11 \cdot 3^{k-2} + 6 \cdot 3^{k-3}) \\
&= 1 - (12 \cdot 2^{k-2} - 11 \cdot 2^{k-2} + 3 \cdot 2^{k-2}) + 2(18 \cdot 3^{k-2} - 11 \cdot 3^{k-2} + 2 \cdot 3^{k-2}) \\
&= 1 - 4 \cdot 2^{k-2} + 2 \cdot 9 \cdot 3^{k-2} = 1 - 2^k + 2 \cdot 2^k
\end{aligned}
$$

So $f(k) = 1 - 2^k + 2 \cdot 2^k$, which is what we needed to show.

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(20 points) Use (strong) induction to prove that, for any integer $n \geq 8$, there are non-negative integers p and q such that $n = 3p + 5q$.

Solution: Proof by induction on n.

Base case(s): At $n = 8$, we can chose $p = 1$ and $q = 1$. At $n = 9$, we can chose $p = 3$ and $q = 0$. At $n = 10$, we can chose $p = 0$ and $q = 2$. In all three cases, $n = 3p + 5q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that there are non-negative integers p and q such that $n = 3p + 5q$, for $n = 8, 9, \ldots, k - 1$, where $k \geq 11$.

Rest of the inductive step: Consider $n = k$.

Notice that $k \geq 11$, so $8 \leq k - 3 \leq k - 1$. So $k - 3$ is covered by the inductive hypothesiss. Therefore, there are non-negative integers r and q such that $k - 3 = 3r + 5q$.

Now, set $p = r + 1$. Then $k = (k - 3) + 3 = (3r + 5q) + 3 = 3(r + 1) + 5q = 3p + 5q$. p is non-negative since r is.

So there are non-negative integers p and q such that $k = 3p + 5q$, which is what we needed to prove.