(15 points) Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: \(n^3 + 5n\) is divisible by 6, for all positive integers \(n\).

**Solution:** Proof by induction on \(n\).

**Base case(s):** At \(n = 1\), \(n^3 + 5n = 6\), which is clearly divisible by 6.

**Inductive Hypothesis** [Be specific, don’t just refer to “the claim”]: Suppose that \(n^3 + 5n\) is divisible by 6, for \(n = 1, 2, \ldots, k\).

**Rest of the inductive step:** Notice that

\[
(k + 1)^3 + 5(k + 1) = (k^3 + 3k^2 + 3k + 1) + (5k + 5) = (k^3 + 5k) + (3k^2 + 3k) + 6 = (k^3 + 5k) + 3k(k + 1) + 6
\]

\((k^3 + 5k)\) is divisible by 6 by the inductive hypothesis. \(3k(k + 1)\) is divisible by 6 because one of \(k\) and \((k+1)\) must be even. 6 is obviously divisible by 6. Since \((k + 1)^3 + 5(k + 1)\) is the sum of these three terms, it must also be divisible by 6, which is what we needed to show.
(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer \( n \), \( \sum_{p=1}^{n} \log(p^2) = 2 \log(n!) \)

Solution: Proof by induction on \( n \).

Base case(s): At \( n = 1 \), \( \sum_{p=1}^{1} \log(p^2) = \log(1^2) = \log 1 = 0 \). Also \( 2 \log(n!) = 2 \log 1 = 2 \cdot 0 = 0 \). So the two are equal.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:

Suppose that \( \sum_{p=1}^{n} \log(p^2) = 2 \log(n!) \) for \( n = 1, \ldots, k \).

Rest of the inductive step: In particular, \( \sum_{p=1}^{k} \log(p^2) = 2 \log(k!) \). Then
\[
\sum_{p=1}^{k+1} \log(p^2) = \log((k+1)^2) + \sum_{p=1}^{k} \log(p^2) = 2 \log(k+1) + \sum_{p=1}^{k} \log(p^2)
\]
\[
= 2 \log(k+1) + 2 \log(k!) = 2(\log(k+1) + \log(k!))
\]
\[
= 2 \log((k+1)!) = 2 \log((k+1)!) \]

So \( \sum_{p=1}^{k+1} \log(p^2) = 2 \log((k+1)!) \), which is what we needed to show.
(15 points) Let \( A \) be a constant integer. Use (strong) induction to prove the following claim. Remember that \( 0! = 1 \).

**Claim:** For any integer \( n \geq A \),

\[
\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!((n-A))!}
\]

**Solution:** Proof by induction on \( n \).

**Base case(s):** At \( n = A \),

\[
\sum_{p=A}^{A} \frac{p!}{A!(p-A)!} = \frac{A!}{A!0!} = 1 = \frac{(A+1)!}{(A+1)!0!} = \frac{(n+1)!}{(A+1)!((n-A))!}
\]

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:

Suppose that \( \sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!((n-A))!} \) is true for all \( n = A, \ldots, k \).

**Rest of the inductive step:**

In particular,

\[
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \frac{(k+1)!}{(A+1)!((k-A))!}.
\]

So then

\[
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \sum_{p=A}^{k} \frac{p!}{A!(p-A)!} + \frac{(k+1)!}{A!(k+1-A)!}
\]

\[
= \frac{(k+1)!}{(A+1)!((k-A))!} + \frac{(k+1)!}{A!(k+1-A)!}
\]

\[
= \frac{(k+1)!}{(A+1)!((k+1-A))!} + \frac{(A+1)(k+1)!}{(A+1)!((k+1-A))!}
\]

\[
= \frac{(k+2)(k+1)!}{(A+1)!((k+1-A))!} = \frac{(k+2)!}{(A+1)!((k+1-A))!},
\]

which is what we needed to show.
The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^5 (p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

$$\prod_{p=2}^n (1 - \frac{1}{p^2}) = \frac{n+1}{2n} \text{ for any integer } n \geq 2.$$

**Solution:** Proof by induction on $n$.

**Base case(s):** At $n = 2$, $\prod_{p=2}^n (1 - \frac{1}{p^2}) = (1 - \frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

**Inductive Hypothesis** [Be specific, don’t just refer to “the claim”]: Suppose that $\prod_{p=2}^n (1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

**Rest of the inductive step:** In particular, from the inductive hypothesis $\prod_{p=2}^k (1 - \frac{1}{p^2}) = \frac{k+1}{2k}$. So

$$\prod_{p=2}^{k+1} (1 - \frac{1}{p^2}) = \left(\prod_{p=2}^k (1 - \frac{1}{p^2})\right)(1 - \frac{1}{(k+1)^2})$$

$$= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2 - 1}{2k(k+1)} = \frac{k^2 + 2k}{2k(k+1)} = \frac{k+2}{2(k+1)}$$

So $\prod_{p=2}^{k+1} (1 - \frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.
(15 points) Use (strong) induction to prove the following claim:

Claim: for all natural numbers \( n \),
\[
\sum_{j=0}^{n} 2(-7)^j = \frac{1 - (-7)^{n+1}}{4}
\]

Solution: Proof by induction on \( n \).

Base case(s): At \( n = 0 \), \( \sum_{j=0}^{n} 2(-7)^j = 2 \) and \( \frac{1-(-7)^{n+1}}{4} = \frac{1-(-7)}{4} = 2 \). So the claim holds at \( n = 0 \).

Inductive Hypothesis [Be specific, don’t just refer to “the claim”]:

Suppose that \( \sum_{j=0}^{n} 2(-7)^j = \frac{1 - (-7)^{n+1}}{4} \) for \( n = 0, 1, \ldots, k \).

Rest of the inductive step:

In particular \( \sum_{j=0}^{k} 2(-7)^j = \frac{1 - (-7)^{k+1}}{4} \). So then

\[
\sum_{j=0}^{k+1} 2(-7)^j = (\sum_{j=0}^{n} 2(-7)^j) + 2(-7)^{k+1} = \frac{1 - (-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1 - (-7)^{k+1} + 8(-7)^{k+1}}{4} = \frac{1 + 7(-7)^{k+1}}{4} = \frac{1 - (-7)^{k+2}}{4}
\]

So \( \sum_{j=0}^{k+1} 2(-7)^j = \frac{1 - (-7)^{k+2}}{4} \), which is what we needed to show.
(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer \( n \), \( \sum_{p=1}^{n} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \sqrt{n} \)

Solution: Proof by induction on \( n \).

Base case(s): At \( n = 1 \), \( \sum_{p=1}^{n} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \frac{1}{\sqrt{1-1} + \sqrt{1}} = \frac{1}{\sqrt{0} + \sqrt{1}} = 1 \) and \( \sqrt{n} = \sqrt{1} = 1 \). So the claim holds.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:
Suppose that \( \sum_{p=1}^{n} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \sqrt{n} \) for \( n = 1, \ldots, k \).

Rest of the inductive step: In particular, \( \sum_{p=1}^{k} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \sqrt{k} \). So then

\[
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \frac{1}{\sqrt{k+\sqrt{k+1}}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \frac{1}{\sqrt{k+\sqrt{k+1}}} + \sqrt{k}
\]

\[
= \frac{\sqrt{k} - \sqrt{k+1}}{k-(k+1)} + \sqrt{k} = \frac{\sqrt{k} - \sqrt{k+1}}{-1} + \sqrt{k}
\]

\[
= (\sqrt{k+1} - \sqrt{k}) + \sqrt{k} = \sqrt{k+1}
\]

So \( \sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1} + \sqrt{p}} = \sqrt{k+1} \), which is what we needed to show.
(15 points) Use (strong) induction to prove the following claim:

Claim: \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \) for all positive integers \( n \).

Solution:

Proof by induction on \( n \).

Base case(s): \( n = 1 \). At \( n = 1 \), \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \). Also \( \frac{n}{n+1} = \frac{1}{2} \). So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]: Suppose that \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \) for \( n = 1, \ldots, k \) for some integer \( k \geq 1 \).

Rest of the inductive step:

Consider \( \sum_{j=1}^{k+1} \frac{1}{j(j+1)} \). By removing the top term of the summation and then applying the inductive hypothesis, we get

\[
\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \frac{k}{k+1}.
\]

Adding the two fractions together:

\[
\frac{1}{(k+1)(k+2)} + \frac{k}{k+1} = \frac{1}{(k+1)(k+2)} + \frac{k(k+2)}{(k+1)(k+2)} = \frac{1+k(k+2)}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}
\]

So \( \sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{k+2} \), which is what we needed to show.
Use (strong) induction and the fact that \( \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \) to prove the following claim:

For all natural numbers \( n \), \( (\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3 \)

**Solution:** Proof by induction on \( n \).

**Base case(s):** At \( n = 0 \), \( (\sum_{i=0}^{n} i)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3 \). So the claim is true.

**Inductive Hypothesis** [Be specific, don’t just refer to “the claim”]:
Suppose that \( (\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3 \) for \( n = 0, 1, \ldots, k \).

**Rest of the inductive step:**
Starting with the lefthand side of the equation for \( n = k + 1 \), we get

\[
\left( \sum_{i=0}^{k+1} i \right)^2 = \left( (k+1) + \sum_{i=0}^{k} i \right)^2 = (k+1)^2 + 2(k+1) \sum_{i=0}^{k} i + \left( \sum_{i=0}^{k} i \right)^2
\]

By the inductive hypothesis \( (\sum_{i=0}^{k} i)^2 = \sum_{i=0}^{k} i^3 \). Substituting this and the fact we were told to assume, we get

\[
\left( \sum_{i=0}^{k+1} i \right)^2 = (k+1)^2 + 2(k+1) \frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3
\]

So \( (\sum_{i=0}^{k+1} i)^2 = \sum_{i=0}^{k+1} i^3 \) which is what we needed to show.