1. (10 points) Suppose that $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is onto. Let’s define $g : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ by $g(x, y) = f(x - 7)f(y)$. Prove that $g$ is onto.

Solution: Suppose that $n$ is an integer.

Since $f$ is onto, there is an integer $p$ such that $f(p) = 1$. Let $x = p + 7$. Then $f(x - 7) = f(p) = 1$.

Also since $f$ is onto, there is a natural number $y$ such that $f(y) = n$.

Now consider the pair $(x, y)$. $g(x, y) = f(x - 7)f(y) = 1 \cdot n = n$. So $(x, y)$ is a pre-image for $n$, which is what we needed to find.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \rightarrow M$ to be “onto.” You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

Solution: For every element $y$ in $M$, there is an element $x$ in $C$ such that $g(x) = y$. 
1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let’s define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (2f(x) + f(y), f(x) - f(y))$. Prove that $g$ is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

**Solution:** Let $(x, y)$ and $(p, q)$ be elements of $\mathbb{Z}^2$ and suppose that $g(x, y) = g(p, q)$.

By the definition of $h$, this means that $(2f(x) + f(y), f(x) - f(y)) = (2f(p) + f(q), f(p) - f(q))$.

So $2f(x) + f(y) = 2f(p) + f(q)$ and $f(x) - f(y) = f(p) - f(q)$.

Adding these two equations, we get $3f(x) = 3f(p)$. So $f(x) = f(p)$. Since $f$ is one-to-one, this means that $x = p$.

Subtracting twice the second equation from the first, we get $-3f(y) = -3f(q)$. So $f(y) = f(q)$.

Since $f$ is one-to-one, this means that $y = q$.

Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be “one-to-one.” You must use explicit quantifiers; do not use words like “unique”.

**Solution:** For every elements $x$ and $y$ in $C$, if $g(x) = g(y)$, then $x = y$
1. (10 points) Suppose that $A$ and $B$ are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that $f$ is onto.

Solution: Let $m$ be an element of $A$. We need to find a pre-image for $m$.

Consider $n = g(m)$. $n$ is an element of $B$. Furthermore, since $f(g(x)) = x$ for every $x \in A$, we have $f(n) = f(g(m)) = m$.

So $n$ is a pre-image of $m$.

Since we can find a pre-image for an arbitrarily chosen element of $A$, $f$ is onto.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto. Your answer must include a specific formula.

Solution: Let $f(n) = n + 1$. Then $f$ is one-to-one, but 0 isn’t in the image of $f$. 


1. (10 points) Suppose that \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) is onto. Let’s define \( g : \mathbb{Z}^2 \rightarrow \mathbb{Z} \) by \( g(x, y) = f(x) + 2f(y) - 6 \). Prove that \( g \) is onto.

**Solution:** Let \( n \) be an arbitrary integer.

Since \( f \) is onto, there is an integer input value \( y \) such that \( f(y) = 3 \). Similarly, there is an integer input value \( x \) such that \( f(x) = n \).

Now, consider \( (x, y) \). \( g(x, y) = f(x) + 2f(y) - 6 = n + 2 \cdot 3 - 6 = n \). So \( (x, y) \) is a pre-image for \( n \), which is what we needed to find.

2. (5 points) \( A = \{0, 2, 4, 6, 8, 10, 12, \ldots\} \), i.e. the even integers starting with 0.

\( B = \{1, 4, 9, 16, 25, 36, 49, \ldots\} \), i.e. perfect squares starting with 1.

Give a specific formula for a bijection \( f : A \rightarrow B \). (You do not need to prove that it is a bijection.)

**Solution:** \( f(n) = \left( \frac{n}{2} + 1 \right)^2 \)
1. (10 points) Suppose that \( f : A \to B \) and \( g : B \to C \) are one-to-one. Prove that \( g \circ f \) is one-to-one.

**Solution:** Let \( x \) and \( y \) be elements of \( A \) and suppose that \( g \circ f(x) = g \circ f(y) \). That is, \( g(f(x)) = g(f(y)) \). Since \( g \) is one-to-one, this implies that \( f(x) = f(y) \). Since \( f \) is one-to-one, this implies that \( x = y \).

We've shown that \( g \circ f(x) = g \circ f(y) \) implies \( x = y \) for any \( x \) and \( y \) in \( A \). So \( g \circ f \) is one-to-one.

2. (5 points) Using precise mathematical words and notation, define what it means for a function \( g : \mathbb{R} \to \mathbb{R} \) to be “increasing.” You must use explicit quantifiers.

**Solution:** For all \( x \) and \( y \) in \( \mathbb{R} \), if \( x \leq y \) then \( g(x) \leq g(y) \).
1. (10 points) Let $P$ be the set of pairs of positive integers. Suppose that $f : P \rightarrow \mathbb{R}^2$ is defined by $f(x, y) = \left(\frac{x}{y}, x + y\right)$. Prove that $f$ is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

**Solution:** Let $(x, y)$ and $(p, q)$ be elements of $P$, i.e. pairs of positive integers. Suppose that $f(x, y) = f(p, q)$.

By the definition of $f$, this means that $\left(\frac{x}{y}, x + y\right) = \left(\frac{p}{q}, p + q\right)$. So $\frac{x}{y} = \frac{p}{q}$ and $x + y = p + q$.

Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{pq}{q}$. Substituting this into $x + y = p + q$ gives us $\frac{pq}{q} + y = p + q$. So $\frac{pq + yq}{q} = p + q$.

I.e. $\frac{y(p + q)}{q} = p + q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

Substituting $y = q$ into $x + y = p + q$ gives us $x + q = p + q$, so $x = p$.

Therefore $(x, y) = (p, q)$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is onto but not one-to-one. Your answer must include a specific formula.

**Solution:** Let $f(n) = \lfloor n/2 \rfloor$ Then $f$ is onto. But $f$ isn’t one-to-one because (for example) both 0 and 1 are mapped onto 0.
1. (10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = 3x + 5y$. Prove that $f$ is onto.

Solution: Let $p$ be an integer. We need to find a pre-image for $p$.

Consider $m = (-3p, 2p)$.

$m$ is an element of $\mathbb{Z}^2$. We can compute

$$f(m) = f(-3p, 2p) = 3(-3p) + 5(2p) = -9p + 10p = p$$

So $m$ is a pre-image of $p$.

Since we can find a pre-image for an arbitrarily chosen integer, $f$ is onto.

2. (5 points) $A = \{0, 1, 4, 9, 16, 25, 36, \ldots \}$, i.e. perfect squares starting with 0.

$B = \{2, 4, 6, 8, 10, 12, 14, \ldots \}$, i.e. the even integers starting with 2.

Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

Solution: $f(n) = 2(\sqrt{n} + 1)$
1. (10 points) Let’s define the function \( f : \mathbb{R}^+ \to \mathbb{R} \) by \( f(x) = \frac{4x - 1}{2x + 5} \). (\( \mathbb{R}^+ \) is the positive reals.)

Prove that \( f \) is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

**Solution:** Let \( x \) and \( y \) be positive reals. Suppose that \( f(x) = f(y) \). By the definition of \( f \), this means that \( \frac{4x - 1}{2x + 5} = \frac{4y - 1}{2y + 5} \).

Multiplying by the two denominators gives us \( (4x - 1)(2y + 5) = (4y - 1)2x + 5 \). That is \( 8xy - 2y + 20x - 5 = 8xy - 2x + 20y - 5 \). So \(-2y + 20x = -2x + 20y\). So \(22x = 22y\). And therefore \( x = y \), which is what we needed to prove.

2. (5 points) Using precise mathematical words and notation, define what it means for a function \( g : \mathbb{R} \to \mathbb{R} \) to be “strictly increasing.” You must use explicit quantifiers.

**Solution:** For all \( x \) and \( y \) in \( \mathbb{R} \), if \( x < y \) then \( g(x) < g(y) \).