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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

(15 points) Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For all integers p and q, if $p^2(q^2-4)$ is odd, then p and q are odd.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all integers p and q, if p is even or q is even, then $p^2(q^2-4)$ is even.

Let p and q be integers. Suppose that p is even or q is even.

There are two cases:

Case (1) p is even. Then there is an integer m such that $p = 2m$. So $p^2(q^2-4) = 4m(q^2-4) = 2(2mq^2 - 8m)$. $t = 2mq^2 - 8m$ is an integer, since m and q are integers. So $p^2(q^2-4) = 2t$ is even.

Case (2) q is even. Then there is an integer n such that $q = 2n$. So $p^2(q^2-4) = p^2(4n^2-4) = 2(2n^2p^2 - 2p^2)$. $r = 2n^2p^2 - 2p^2$ is an integer because n and p are integers. So $p^2(q^2-4) = 2r$ is even.

In both cases, $p^2(q^2-4)$ is even, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

> For all integers x, y, p, q and m, with $m > 0$, if $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$, then $x^2 + xy \equiv p^2 + pq \pmod{m}$.

Solution: Let x, y, p, q and m be integers, with $m > 0$. Suppose that $x \equiv p \pmod{m}$ and $y \equiv q \pmod{m}$.

By the definition of congruence mod k, this means that $x = p + am$ and $y = q + bm$, for some integers a and b. Then we can calculate

$$\begin{aligned}
x^2 + xy &= (p + am)^2 + (p + am)(q + bm) \\
&= (p + am)(p + am + q + bm) \\
&= (p + am)(p + q) + (p + am)(am + bm) \\
&= (p + am)(p + q) + m(p + am)(a + b)
\end{aligned}$$

Let $t = (p + am)(a + b)$. Then we have

$$\begin{aligned}
x^2 + xy &= (p + am)(p + q) + mt \\
&= p(p + q) + am(p + q) + mt = p^2 + pq + m(ap + aq + t)
\end{aligned}$$

$(ap + aq + t)$ is an integer because a, b, m, p, q are all integers. So $x^2 + xy \equiv p^2 + pq \pmod{m}$.

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(15 points) For any two real numbers x and y, the harmonic mean is $H(x, y) = \frac{2xy}{x+y}$. Using this definition and your best mathematical style, prove the following claim:

For any real numbers x and y, if $0 < x < y$, then $H(x, y) < y$.

Solution: Let x and y be real numbers. Suppose that $0 < x < y$.

Since $x < y$ and y is positive, $xy < y^2$.

Adding xy to both sides gives us $2xy < xy + y^2 = y(x + y)$.

So $2xy < y(x+y)$. Since x and y are both positive, $x+y$ is positive. So, we can divide both sides by $(x + y)$ to get $\frac{2xy}{x+y} < y$.

So, $H(x, y) < y$, which is what we needed to prove.

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(15 points) Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim by contrapositive.

> For all real numbers x and y, if x is not rational, then $2x + 3y$ is not rational or y is not rational.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if $2x+3y$ is rational and y is rational, then x is rational.

Let x and y be real numbers. Suppose that $2x + 3y$ is rational and y is rational. Then $2x + 3y = \frac{a}{b}$ and $y = \frac{m}{n}$, where a, b, m, n are integers, b and n non-zero.

Then $2x + 3\frac{m}{n} = \frac{a}{b}$

So $2x = \frac{a}{b} - \frac{3m}{n} = \frac{an-3bm}{bn}$

So $x = \frac{an-3bm}{2bn}$

$an - 3bm$ and $2bn$ are both integers because a, b, m, n are integers. Also $2bn$ is non-zero because b and n are non-zero. So x is rational.

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(15 points) A triple (a, b, c) of positive integers is Pythagorean if $a^2 + b^2 = c^2$. Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For any Pythagorean triple (a, b, c), if c^2 is odd, then a is even or b is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for any Pythagorean triple (a, b, c), if a and b are odd, then c^2 is even.

So suppose (a, b, c) is Pythagorean and a and b are odd. Then $a^2 + b^2 = c^2$ by the definition of Pythagorean. Also, by the definition of odd, $a = 2m + 1$ and $b = 2p + 1$, where m and n are integers.

Then $c^2 = a^2+b^2 = (2m+1)^2+(2p+1)^2 = (4m^2+4m+1)+(4p^2+4p+1) = 2(2m^2+2m+2p^2+2p+1)$

Let $t = 2m^2 + 2m + 2p^2 + 2p + 1$. t is an integer because m and p are integers. And $c^2 = 2t$. So c^2 is even.

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(15 points) For any two real numbers x and y, the harmonic mean is $H(x,y) = \frac{2xy}{x+y}$. Using this definition and your best mathematical style, prove the following claim:

For any real numbers x and y, if $0 < x < y$, then $x < H(x,y)$.

Solution: Let x and y be real numbers. Suppose that $0 < x < y$.

Since $x < y$ and x is positive, $x^2 < xy$.

Adding xy to both sides gives us $x^2 + xy < 2xy$.

So $x(x+y) < 2xy$. Since x and y are both positive, $x+y$ is positive. So, we can divide both sides by $(x+y)$ to get $x < \frac{2xy}{x+y}y$.

So, $x < H(x,y)$, which is what we needed to prove.

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(15 points) Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $x \equiv y \pmod{k}$ if and only if $x = y + nk$ for some integer n.

For all integers a, b, c, p and k (c positive), if $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$, then $k \mid b$.

Solution:

Let a, b, c, p and k be integers, with c positive. Suppose that $ap \equiv b \pmod{c}$ and $k \mid a$ and $k \mid c$.

By the definition of congruence mod k, $ap \equiv b \pmod{c}$ implies that $ap = b + nc$ for some integer n. By the definition of divides, $k \mid a$ and $k \mid c$ imply that $a = ks$ and $c = kt$ for some integers s and t.

Since $ap = b + nc$, $b = ap - nc$. So then we have

$b = ap - nc = ksp - nkt = k(sp - nt)$

$sp - nt$ is an integer since s, p, n, and t are integers. So this implies that $k \mid b$.

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(15 points) Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases.

For any integer n, $n^2 + 2$ is not divisible by 4.

Solution: Let n be an integer. From the Division Algorithm (aka definition of remainder), we know that there are integers q and r such that $n = 4q + r$.

There are four cases, depending on what the remainder r is:

Case 1: $n = 4q$. Then $n^2 + 2 = 16q^2 + 2 = 4(4q^2) + 2$.

Case 2: $n = 4q + 1$. Then $n^2 + 2 = 16q^2 + 8q + 3 = 4(4q^2 + 2q) + 3$.

Case 3: $n = 4q + 2$. Then $n^2 + 2 = 16q^2 + 16q + 6 = 4(4q^2 + 4q + 1) + 2$.

Case 4: $n = 4q + 3$. Then $n^2 + 2 = 16q^2 + 24q + 11 = 4(4q^2 + 6q + 2) + 3$.

In all four cases, the remainder of $n^2 + 2$ divided by 4 is not zero, so $n^2 + 2$ isn't divisible by 4.