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Let $f : \mathbb{Z}_{12} \to \mathbb{P}(\mathbb{Z}_{12})$ be defined by $f(x) = \{y \in \mathbb{Z}_{12} \mid y^2 = x\}$. Let $S = \{f(x) \mid x \in \mathbb{Z}_{12}\}$.

(3 points) $S =$

Solution: $\{\{2, 4, 8, 10\}, \{0, 6\}, \{1, 5, 7, 11\}, \{3, 9\}, \emptyset\}$

(3 points) Is S a partition of $\mathbb{Z}_{12}$? Check the partition properties that are satisfied.

No Empty set ☐ No Partial Overlap ☑ $\surd$ Covers base set ☑ $\surd$

(7 points) Suppose that $A_1, A_2, \ldots, A_n$ are non-empty subsets of A, and let $P = \{A_1, A_2, \ldots, A_n\}$. Also suppose that $A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = A$. Is P a partition of A? Explain why or why not.

Solution: P is not necessarily a partition of A. The issue is that $A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset$ can be true even when some pairs of (distinct) subsets overlap. For example, $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, and $A_3 = \{3, 4\}$. Then $A_1 \cap A_2 \cap A_3 = \emptyset$ but A_1 and A_2 intersect.

(2 points) Check the (single) box that best characterizes each item.

If $f : \mathbb{R} \to \mathbb{P}(\mathbb{Z})$ then $f(17)$ is	an integer ☐	a set of integers ☑ $\surd$	undefined ☐
	a power set ☐	one or more integers ☐	

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(7 points) Suppose that A is a set and P is a collection of subsets of A. Using precise language and/or notation, state the conditions P must satisfy to be a partition of A.

Solution: P cannot contain the empty set. Every element of A must belong to exactly one element of P.

The second condition is frequently split into two sepaate conditions. That is, every element of A must belong some to element of P, and two distinct elements of P cannot overlap.

(2 points) $\{\{p, q\} \ : \ p \in \mathbb{Z}^+, q \in \mathbb{Z}^+, \text{ and } pq = 6\} \ =$

Solution: $\{\{1, 6\}, \{2, 3\}\}$

(6 points) Check the (single) box that best characterizes each item.

$\{\{a, b\}, c\} = \{a, b, c\}$ — true ☐ false ☑ ($\surd$)

If $f : \mathbb{N} \to \mathbb{P}(\mathbb{Q})$ then $f(3)$ is — a rational ☐ a set of rationals ☑ ($\surd$) undefined ☐ a power set ☐ one or more rationals ☐

$\binom{k}{k-1}$ — 1 ☐ 2 ☐ k-1 ☐ k ☑ ($\surd$) undefined ☐

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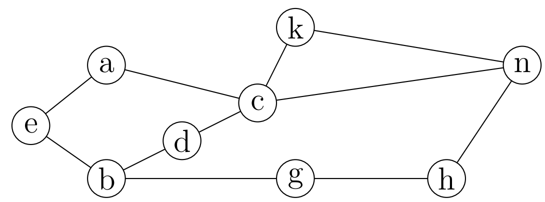
Graph G is at right.
V is the set of nodes. E is the set of edges.



Let $M : (V, \mathbb{N}) \to \mathbb{P}(V)$ such that $M(x, n) = \{y \in V \mid \text{ there is a path of length } n \text{ from } x \text{ to } y\}$.
Let $P(x) = \{M(x, n) \mid n \in \mathbb{N}\}$.

(3 points) $M(c, 2) =$

Solution: $M(c, 2) = \{b, e, n, h, k\}$

(3 points) Is $P(c)$ a partition of V? Check the partition properties that are satisfied.

No Empty set ☐ No Partial Overlap ☐ Covers base set ☑ $\surd$

(7 points) Let $f : X \to Y$ be any function, and let A and B be subsets of X. For any subset S of X define its image $f(S)$ by $f(S) = \{f(s) \in Y \mid s \in S\}$. Is it the case that $f(A) \cap f(B) = f(A \cap B)$? Informally explain why this is true or give a concrete counter-example showing why it is not.

Solution: This is false. Let $X = \{a, b\}$ and $Y = \{c\}$. Define $f : X \to Y$ by $f(x) = c$ for all $x \in X$. Suppose that $A = \{a\}$ and $B = \{b\}$. Then $A \cap B = \emptyset$, $sof(A \cap B) = \emptyset$. But $f(A) \cap f(B) = \{c\}$

(2 points) Check the (single) box that best characterizes each item.

$\{4, 5, 6\} \cap \{6, 7\}$ 6 ☐ $\{6\}$ ☑ $\surd$ $\{\{6\}\}$ ☐

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(7 points) Suppose that $f : A \to B$ is a function. Let's define $T : B \to \mathbb{P}(A)$ by $T(m) = \{x \in A \mid f(x) = m\}$. Then let $P = \{T(m) \mid m \in B\}$. Under what conditions is P a partition of A? Briefly justify your answer.

Solution: $T(m)$ is the set of pre-images of m. Every element $x \in A$ has exactly one image in B. So it belongs to exactly one set $T(m)$. That covers two of the partition properties.

However, P will contain the empty set if f is not onto. So P is a partition if and only if f is onto.

(2 points) $\{p + q^2 \mid p \in \mathbb{Z},\ q \in \mathbb{Z},\ 1 \leq p \leq 2 \text{ and } 1 \leq q \leq 3\} =$

Solution: $\{2, 3, 5, 6, 10, 11\}$

(6 points) Check the (single) box that best characterizes each item.

$\mathbb{P}(A) \cap \mathbb{P}(B) = \emptyset$ always ☐ sometimes ☐ never ☑ $\surd$

Set B is a partition of a finite set A. Then

$|B| \leq 2^{|A|}$ ☐ $|B| \leq |A|$ ☑ $\surd$

$|B| = 2^{|A|}$ ☐ $|B| \leq |A + 1|$ ☐

Pascal's identity states that $\binom{n}{k}$ is equal to

$\binom{n-1}{k} + \binom{n-1}{k-1}$ ☑ $\surd$ $\binom{n-1}{k} + \binom{n-1}{k+1}$ ☐ $\binom{n-1}{k} + \binom{n-2}{k}$ ☐

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Let $f : \mathbb{R} \to \mathbb{P}(\mathbb{R})$ such that $f(x) = \{p \in \mathbb{R} \mid \lfloor x \rfloor = \lfloor p \rfloor\}$. Let $T = \{f(x) \mid x \in \mathbb{R}\}$.

(3 points) Describe (at a high level) the elements of $f(7)$:

Solution: All the real numbers whose floor is 7.

(3 points) Is T a partition of $\mathbb{R}$? Check the partition properties that are satisfied.

No Empty set [√] No Partial Overlap [√] Covers base set [√]

(7 points) Define $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{P}(\mathbb{Z})$ by $f(x, k) = \{y \in \mathbb{Z} : x = y + kn \text{ for some } n \in \mathbb{Z}\}$. Suppose that $k|p$. Compare $f(r, k)$ and $f(r, p)$. Justify your answer.

Solution: $f(r, p)$ is a subset of $f(r, k)$. Because k divides p, two numbers that differ by a multiple of p must also differ by a multiple of k, but not vice versa. So each equivalence class mod k is the union of several equivalence classes mod p.

(2 points) Check the (single) box that best characterizes each item.

$\mathbb{P}(\emptyset)$ $\emptyset$ [ ] $\{\emptyset\}$ [√] $\{\{\emptyset\}\}$ [ ] $\{\emptyset, \{\emptyset\}\}$ [ ]

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(7 points) Let $f : \mathbb{Z}^+ \to \mathbb{P}(\mathbb{Z}^+)$ be defined by $f(n) = \{p \in \mathbb{Z}^+ : n|p\}$. Suppose that $f(a) = f(b) \cap f(c)$. Express a in terms of b and c. Briefly justify your answer.

Solution: Every element of $f(b)$ contains all multiples of b and $f(c)$ contains multiples of c. So $f(a)$ must contain all numbers that are multiples of both b and c. a is the smallest element of $f(a)$. So $a = lcm(b, c)$.

(2 points) $\{\{p\} \mid p \in \{2, 3, 4\}\} =$

Solution: $\{\{2\}, \{3\}, \{4\}\}$

(6 points) Check the (single) box that best characterizes each item.

$\lvert\{A \subseteq \mathbb{Z}_4 : \lvert A\rvert \text{ is even}\}\rvert$	1 ☐	6 ☐	7 ☐	8 ☑	infinite ☐	
There is a set A such that $\lvert\mathbb{P}(A)\rvert \leq 2$.	true ☑	false ☐				
$\binom{n}{1}$	-1 ☐	0 ☐	1 ☐	2 ☐	n ☑	undefined ☐

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Suppose that $A = \{2, 3, 5, 13, 17\}$. Define a function $F : A \to \mathbb{P}(A)$ and a set S by $F(x) = \{y \in A \mid y \text{ is a factor of } x\}$ $\quad S = \{F(x) \mid x \in A\}$

(3 points) $S =$

Solution: $\{\{2\}, \{3\}, \{5\}, \{13\}, \{17\}\}$.

(3 points) Is S a partition of A? Check the partition properties that are satisfied.

No Empty set [√] No Partial Overlap [√] Covers base set [√]

(7 points) Let $f : X \to Y$ be any function, and let A and B be subsets of X. For any subset S of X define its image $f(S)$ by $f(S) = \{f(s) \in Y \mid s \in S\}$. Is it the case that $f(A) \cup f(B) = f(A \cup B)$? Informally explain why this is true or give a concrete counter-example showing why it is not.

Solution: This is true.

y is in $f(A \cup B)$ if and only if y is the image of a value in $A \cup B$. But this is true exactly when y is the image of a value in A or y is the image of a value in B. That is y is in $f(A) \cup f(B)$.

(2 points) Check the (single) box that best characterizes each item.

A partition of a set A contains $\emptyset$ always [ ] sometimes [ ] never [√]

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(7 points) Give an example of a partition P of $\mathbb{N}$ where the set P is infinite. Be specific.

Solution: Suppose that each natural number is in its own partition set. That is $P = \{\{x\} \mid x \in \mathbb{N}\}$. Then P is a partition of $\mathbb{N}$ and P is infinite.

(2 points) $\{pq \mid p \in \mathbb{N},\ q \in \mathbb{N},\ p + q = 6\} =$

Solution: $\{0, 5, 8, 9\}$

(6 points) Check the (single) box that best characterizes each item.

$\mathbb{P}(A) \cup \mathbb{P}(B) = \mathbb{P}(A \cup B)$ always ☐ sometimes ☑ never ☐

$|\{\emptyset\}|$ 0 ☐ 1 ☑ 2 ☐ 3 ☐ 4 ☐ undefined ☐

$\{4, 5\} \cap \{6, 7\}$ $\emptyset$ ☑ $\{\emptyset\}$ ☐ nothing ☐ undefined ☐