1. (9 points) Fill in key facts about the recursion tree for $T$, assuming that $n$ is even.

$$T(0) = 5 \quad T(n) = 3T(n-2) + n^2$$

(a) The height: $\frac{n}{2}$

(b) The number of leaves (please simplify): $3^{\frac{n}{2}} = (\sqrt{3})^n$

(c) Value in each node at level $k$: $(n-2k)^2$

Change of base formula: $\log_b n = \left( \log_a n \right) \left( \log_b a \right)$

2. (6 points) Write the following functions in the boxes so that $f(n) \ll g(n)$.

$$n \quad n \log(17n) \quad \sqrt{n} + 18 \quad 8n^2 \quad 2^n + n! \quad 2^{\log_4 n} + 5^n \quad 0.001n^3 + 3^n$$

Solution:

$$\sqrt{n} + 18 \ll n \ll n \log(17n) \ll 8n^2 \ll 0.001n^3 + 3^n \ll 2^{\log_4 n} + 5^n \ll 2^n + n!$$
1. (7 points) Recall that $f$ is $O(g)$ if and only if there are positive reals $c$ and $k$ such that $0 \leq f(x) \leq cg(x)$ for every $x \geq k$. Prof. Snape claims that there is a function $f$ (from the reals to the reals) that can never be involved in a big-O relationship. Is he correct?

Solution: He is right. Our definition will never be satisfied if one (or both) of the functions produces only negative outputs.

2. (8 points) Check the (single) box that best characterizes each item.

- $T(1) = c$
- $T(n) = 2T(n/2) + n$

- $T(1) = d$
- $T(n) = T(n/2) + c$

- $n^{1.5}$ is

- $n^{\log_5}$ grows

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- $n^{1.5}$ is

- $\Theta(n^{1.614})$ | $O(n^{1.614})$ | neither of these

- $n^{\log_5}$ grows

- faster than $n^2$
- slower than $n^2$
- at the same rate as $n^2$
1. (9 points) Fill in key facts about the recursion tree for $T$, assuming that $n$ is even.

   $T(8) = 5$  
   $T(n) = 3T(n - 2) + c$

   (a) The height: $\frac{n}{2} - 4$
   (b) The number of nodes at level $k$: $3^k$
   (c) Value in each node at level $k$: $c$

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that $f$ is to the left of $g$ if and only if $f(n) \ll g(n)$.

   $3n^2 \quad \frac{n \log n}{7} \quad (10^{10})n \quad 0.001n^3 \quad 30 \log(n^{17}) \quad 8n! + 18 \quad 3^n + 11^n$

   Solution:

   $30 \log(n^{17}) \ll (10^{10})n \ll \frac{n \log n}{7} \ll 3n^2 \ll 0.001n^3 \ll 3^n + 11^n \ll 8n! + 18$
1. (7 points) Suppose that $f$ and $g$ are functions from the reals to the reals. Define precisely what it means for $g$ to be $\Theta(f)$. Your definition can be in terms of other primitives such as $\ll$ and big-O.

**Solution:** $g$ is $\Theta(f)$ if and only if $g$ is $O(f)$ and $f$ is $O(g)$.

2. (8 points) Check the (single) box that best characterizes each item.

\[
\begin{align*}
T(1) &= d \\
T(n) &= T(n-1) + c \\
\Theta(\log n) & \quad \Theta(\sqrt{n}) \\
\Theta(n^2) & \quad \Theta(n^3) \\
\Theta(n) & \quad \sqrt{\Theta(n)} \\
\Theta(n \log n) & \quad \Theta(3^n)
\end{align*}
\]

\[
\begin{align*}
T(1) &= d \\
T(n) &= 2T(n/2) + c \\
\Theta(\log n) & \quad \Theta(\sqrt{n}) \\
\Theta(n^2) & \quad \Theta(n^3) \\
\Theta(n) & \quad \sqrt{\Theta(n)} \\
\Theta(n \log n) & \quad \Theta(3^n)
\end{align*}
\]

Suppose $f$ and $g$ produce only positive outputs and $f(n) \ll g(n)$. Will $g(n)$ be $O(f(n))$?

- no \quad \sqrt{\text{sometimes}} \quad \text{yes}

\[
n^{\log_4 4} \text{ grows faster than } n^2 \quad \text{slower than } n^2 \quad \text{at the same rate as } n^2
\]
1. (9 points) Fill in key facts about the recursion tree for \( T \), assuming that \( n \) is a multiple of 3.

\[
T(3) = 7 \quad T(n) = 2T(n - 3) + c
\]

(a) The height: \( \frac{n}{3} - 1 \)
(b) The number of leaves (please simplify): \( 2^{\frac{n}{3} - 1} \)
(c) Total work (sum of the nodes) at level \( k \) (please simplify): There are \( 2^k \) nodes at level \( k \), each containing value \( c \). So the total work is \( c2^k \).

Change of base formula: \( \log_b n = (\log_a n)(\log_b a) \)

2. (6 points) Write the following functions in the boxes so that \( f \) is to the left of \( g \) if and only if \( f(n) \ll g(n) \).

\[
n \quad n \log(17n) \quad \sqrt{n} + 2^n + 18 \quad 8n^2 \quad 2^n + n! \quad 2^{\log_4 n} \quad 0.001n^3 + 3^n
\]

Solution:

\[
2^{\log_4 n} \ll n \ll n \log(17n) \ll 8n^2 \ll \sqrt{n} + 2^n + 18 \ll 0.001n^3 + 3^n \ll 2^n + n!
\]
1. (7 points) Suppose that \( f, g, \) and \( h \) are functions from the reals to the reals, such that \( f(x) \) is \( O(h(x)) \) and \( g(x) \) is \( O(h(x)) \). Must \( f(x)g(x) \) be \( O(h(x)) \)?

**Solution:** This is false.

Suppose that \( f(x) = g(x) = h(x) = x \). Then \( f(x) \) is \( O(h(x)) \) and \( g(x) \) is \( O(h(x)) \), but \( f(x)g(x) = x^2 \) is not \( O(h(x)) \).

2. (8 points) Check the (single) box that best characterizes each item.

\[
\begin{array}{cccccccc}
T(1) = c & \Theta(\log n) & \Theta(\sqrt{n}) & \Theta(n) & \Theta(n \log n) & \sqrt{ } \\
T(n) = 3T(n/3) + n & \Theta(n^2) & \Theta(n^3) & \Theta(2^n) & \Theta(3^n) & \checkmark \\
\end{array}
\]

\[
\begin{array}{cccccccc}
T(1) = c & \Theta(\log n) & \Theta(\sqrt{n}) & \Theta(n) & \Theta(n \log n) & \checkmark \\
T(n) = 2T(n/2) + n^2 & \Theta(n^2) & \sqrt{ } & \Theta(n^3) & \Theta(2^n) & \Theta(3^n) & \checkmark \\
\end{array}
\]

Dividing a problem of size \( n \) into \( k \) sub-problems, each of size \( n/m \), has the best big-\( \Theta \) running time when

\[
k < m \quad \checkmark \quad k = m \quad \checkmark \quad k > m \quad \checkmark \quad km = 1 \quad \checkmark
\]

\( n^{\log_2 5} \) grows faster than \( n^2 \) \( \checkmark \) slower than \( n^2 \) \( \checkmark \) at the same rate as \( n^2 \) \( \checkmark \)
1. (9 points) Fill in key facts about the recursion tree for $T$, assuming that $n$ is a power of 2.

\[ T(8) = 7 \quad T(n) = 4T\left(\frac{n}{2}\right) + n \]

(a) The height: $\log_2 n - 3$

(b) Total work (sum of the nodes) at level $k$ (please simplify): There are $4^k$ nodes at level $k$. Each one contains the value $\frac{n}{2^k}$. So the total for the level is $2^k n$.

(c) The number of leaves (please simplify): $4^{\log_2 n - 3} = \frac{1}{4^3} 4^{\log_2 n}$

\[ 4^{\log_2 n} = 4^{\log_4 n \log_2 4} = (4^{\log_4 n})^{\log_2 4} = n^{\log_4 4} = n^2 \]

So the number of leaves is $\frac{1}{4^3} n^2$.

Change of base formula: $\log_b n = \left(\frac{\log_a n}{\log_a b}\right)$

2. (6 points) Write the following functions in the boxes so that $f$ is to the left of $g$ if and only if $f(n) \ll g(n)$.

$3^n \quad 4^{\log_2 n} \quad 2^{3n} \quad 3^{\log_4 4} \quad 0.1n \quad (5n)! \quad \sqrt{n}$

Solution:

$3^{\log_4 4} \ll \sqrt{n} \ll 0.1n \ll 4^{\log_2 n} \ll 3^n \ll 2^{3n} \ll (5n)!$
1. (7 points) Suppose that $f$ and $g$ are functions from the reals to the reals. Define precisely when $f \ll g$.

Solution: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

2. (8 points) Check the (single) box that best characterizes each item.

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T(1) = d \\
T(n) = 3T(n-1) + c
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T(1) = d \\
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