CS 173 Section B, Spring 2016, Examlet 5

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

Discussion Section Time on Tue

Problem	1	2	3	4	5	Total
Possible	20	15	10	10	20	75
Score						

1. **[4 + 4 + 8 + 4 = 20 points]**

(a) There are a set of 73 cities in mainland United States and you are in charge of building highways for automatic cars that connect these cities. If each highway connects two cities, how many highways do you minimally need to make sure cars can get from any city to any other city (possibly through other cities)?

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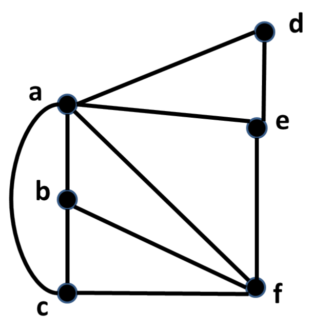
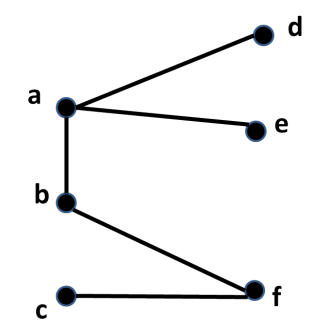
(b) A *spanning tree* of a graph G is a subgraph H of G which is a tree and includes all vertices of G.

For the following graph G, give a spanning tree of the graph.



Solution:



(c) An edge in a connected graph is said to be a *cut-edge* if removing the edge makes the graph disconnected (not connected).

Professor Graphnütter claims that if G is a connected graph such that every edge of G is a cut-edge, then G must be a tree.

Either argue that the claim is correct or give a counterexample.

Solution:
The claim is correct. Let G be a connected graph such that every edge of G is a cut-edge. Then G must be acyclic.
Proof: Assume it is not. Consider a cycle in G and an edge $\{u, v\}$ in the cycle. Now, if we remove this edge and get the graph G', the graph G' will still be connected, since in any path between two vertices in G that used the edge $\{u, v\}$, we can create a path between the same vertices in v' where we replace the edge $\{u, v\}$ with the rest of the cycle that will connect u and v. Hence $\{u, v\}$ cannot be a cut-edge. And the contradiction proves the claim.

(d) Let $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(0) = 3$$
$$f(1) = 2$$
$$f(n) = n + f(n-2) + 2f(n-1) \textit{ for every } n \geq 3$$

What is $f(5)$?
Solution:

$$f(2) = 2 + f(0) + 2f(1) = 2 + 3 + 4 = 9$$

$$f(3) = 3 + f(1) + 2f(2) = 3 + 2 + 18 = 23$$

$$f(4) = 4 + f(2) + 2f(3) = 4 + 9 + 46 = 59$$

$$f(5) = 5 + f(3) + 2f(4) = 5 + 23 + 118 = 146$$

2. **[15 points]**

Recall the Fibonacci sequence: $F(0) = 0$, $F(1) = 1$, and every $i > 1$, $F(i) = F(i-1) + F(i-2)$. Prove that for every $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} F(i)^2 = F(n) \cdot F(n+1)$$

Solution: We will prove by induction on n that for every $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} F(i)^2 = F(n) \cdot F(n+1)$$

Base case: n=0 When $n = 0$,

$$\sum_{i=0}^{n} F(i)^2 = F(0)^2 = 0$$

and $F(n)F(n+1) = F(0)F(1) = 0$. Hence the claim is correct when $n = 0$.

Induction step:
Let $n > 0$.
Assume the *induction hypothesis*:

For every $0 \leq j < n$,

$$\sum_{i=0}^{j} F(i)^2 = F(j) \cdot F(j+1)$$

Now,

$$\begin{aligned}
\sum_{i=0}^{n} F(i)^2 &= \sum_{i=0}^{n-1} F(i)^2 \quad + \quad F(n)^2 \\
&= F(n-1) \cdot F(n) \quad + \quad F(n)^2 \qquad \textit{(by the induction hypothesis, since } 0 \leq n-1 < n\textit{)} \\
&= F(n)(F(n-1) + F(n)) \\
&= F(n)F(n+1)
\end{aligned}$$

which proves the claim for n. QED.

3. **[10 points]**
Recall that a rational number is a real number that can be expressed as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, and $q \neq 0$. And an irrational number is a real number that is not rational. You can assume that the sum/difference/product of two rational numbers is rational.

Prove that for any two real numbers x, y, if x is rational and y is irrational, then $x+y$ is irrational.

Hint: Use a proof by contradiction.

Solution:

Assume the contrary, i.e., assume that there are two real numbers x, y such that x is rational, y is irrational, and $x + y$ is rational.

Then $y = (x + y) - x$ and since the difference of two rationals is rational, y must be rational.
But this contradicts our assumption that y is irrational.
This contradiction proves that our assumption was wrong, and hence the claim is true. QED.

4. **[10 points]**
Let $A = \{5a + 3b \mid a, b \in \mathbb{Z}\}$.
Prove that $\mathbb{Z} \subseteq A$.
Solution:
Let $x \in \mathbb{Z}$ be an arbitrary integer.
Let $a = 2x$ and $b = -3x$. Clearly $a, b \in \mathbb{Z}$.
Hence $5a + 3b = 10x - 9x = x$ belongs to A.
Hence $\mathbb{Z} \subseteq A$. QED.

5. **[20 points]**

A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is said to be a *chooser function* if it satisfies the following:

For every $x, y \in \mathbb{N}$, $f(x, y) = x$ or $f(x, y) = y$, and
for every $x, y \in \mathbb{N}$, $f(x, y) = f(y, x)$.

For example, the function:

$g(x, y) = max(x, y)$

(which returns the maximum of x and y) is a chooser function.

Let f be a chooser function. A full binary rooted tree T where nodes are labeled with natural numbers is said to be an f-tree if the following holds:

- For every internal node n in the tree, if the label on its two children are x and y, then the label on n is $f(x, y)$.

For any chooser function f and for any f-tree T, prove that the label of the root of T is the same as the label of one of its leaves.

Solution: Let f be an arbitrary chooser function.
We will prove by induction on h that for every $h \in \mathbb{N}$ that
for any f-tree of height h, the label of its root is the same as the label of one of its leaves.

Base case: $h = 0$ When $h = 0$, the tree is a single node that is both the root and a leaf, and hence the label of the root is the same as the label of a leaf.

Induction step:
Let $h > 0$ and consider an arbitrary f-tree T of height h.

Assume the *induction hypothesis*:

> For any f-tree of height j, where $0 \leq j < h$, the label of its root is the same as the label of one of its leaves.

Since the height of T is greater than 0, the root of T must have a child, and since T is a full binary tree, the root must have two children.

Consider the two trees T_l and T_r that are rooted at the left child and the right child of the root, respectively.
The heights of these trees is strictly less than h.
Moreover, notice that these trees are themselves f-trees— they are clearly full binary trees, and they satisfy the condition that the label of any internal node is the value obtained by applying f to the label of its children.
Let l_1 and l_2 be the labels of the left child and right child of the root (i.e., the labels of the roots of T_l and T_r).
Then, by the induction hypothesis, l_1 matches the label of a leaf of T_l and l_r matches the label of

a leaf of T_r.
Now the label of the root of T is, by definition of f-trees, equal to $f(l_1, l_2)$, and since f is a chooser function, the label of the root of T is either l_1 or l_2.
In either case, this label matches the label of some leaf of T, which proves the claim for T.

Hence we have proved by induction that the label of the root of any f-tree must be the same as the label of one of its leaves. QED.