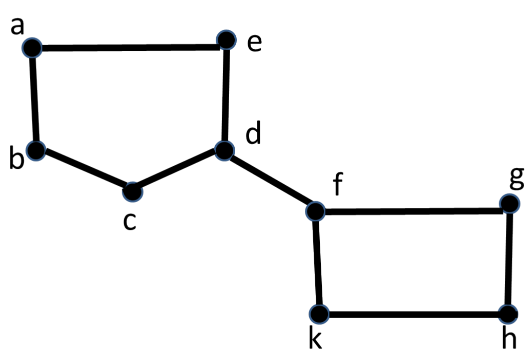
CS 173 Section B, Spring 2016, Examlet 4

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

Problem	1	2	3	4	5	Total
Possible	5	16	14	15	15	65
Score						

1. [**5 points**]

Give all the automomorphisms for this graph.
(An automorphism is an isomorphism between a graph and itself.)



Solution: Let $V = \{a, b, c, d, e, f, g, h, k\}$.

There are four automorphisms for this graph:

The identity isomorphism:
$m(v) = v$ for all $v \in V$.

The isomorphism:
$m(c) = e, m(b) = a, m(a) = b, m(e) = c$, and for every $v \in \{d, f, g, h, k\}$, $m(v) = v$.

The isomorphism:
$m(k) = g, m(g) = k$, and for every $v \in \{a, b, c, e, d, f, h\}$, $m(v) = v$.

The isomorphism:
$m(c) = e, m(b) = a, m(a) = b, m(e) = c, m(k) = g, m(g) = k$, and for every $v \in \{d, f, h\}$, $m(v) = v$.

2. **Short questions [16 points]**

(a) How many edges does a (free) tree with n nodes have?

$n - 1$ edges

(b) Let u, v be two nodes in a tree. Then there is a path from u to v

True [X] False [ ]

(c) Let u, v be two nodes in a tree. Then there can be two paths (with no nodes repeating) from u to v.

True [ ] False [X]

(d) There exists a tree T and an edge in it such that its removal results in a tree as well.

True [ ] False [X]

(e) There is a tree T and two vertices in T (that are not connected by an edge) such that adding an edge between these vertices results in a tree as well.

True [ ] False [X]

(f) Let G be a connected graph and assume that removing any edge from G makes the graph disconnected. Then G is a tree.

True [X] False [ ]

(g) A full m-ary rooted tree is a rooted tree where every node either has no children or has m children. If a full m-ary tree has t internal nodes, then how many leaves does it have?

$mt + 1 - t = t(m - 1) + 1$

(h) What is the minimum number of colors required to color *any* tree?

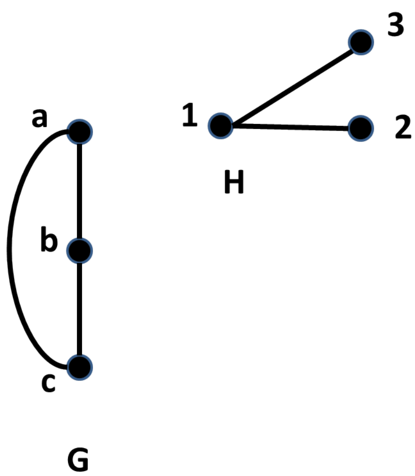
2

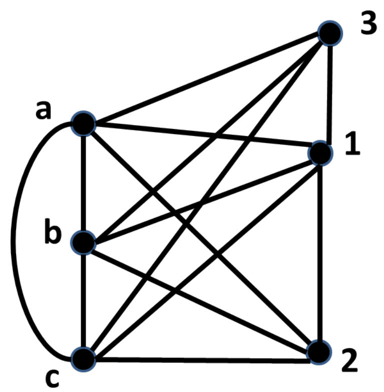
3. **[4+10=14 points]**

For any two graphs G and H with disjoint vertex sets, let us define the *paired-graph of* G *and* H to be the graph obtained by taking G and H together, and adding edges between every vertex of G and every vertex of H.

More formally, if $G = (V_1, E_1)$ and $H = (V_2, E_2)$, with $V_1 \cap V_2 = \emptyset$, then the paired-graph of G and H is the graph $(V_1 \cup V_2, E)$ where $E = E_1 \cup E_2 \cup \{vv' \mid v \in V_1, v' \in V_2\}$.

(a) Give any two graphs G and H (G and H should be different, and have at least two vertices and two edges each). And give the paired-graph of G and H.





The paired graph of G and H

(b)
Recall: the chromatic number of a graph is the minimum number of colors required to color it.

Assume that G and H are two graphs with disjoint sets of vertices, and assume that G has chromatic number r and H has chromatic number s.

Then what is the chromatic number of the paired-graph of G and H?

You are required to give the chromatic number of the paired-graph in terms of r and s (cross-check it against your example above!), argue that the paired-graph can be colored with the number of colors you claim, and argue that the paired-graph cannot be colored with any fewer colors than what you claim.

Solution:

The chromatic number of the paired graph of G and H is $r + s$.

We can show that $r + s$ colors is sufficient to color the paired graph:
Color the vertices of G using r colors and color the vertices of H using a *different* set of s colors. This is a valid $r + s$-coloring as clearly every edge in G and H will be incident on vertices with different colors, and every new edge added in the paired graph will also connect vertices with different colors.

We can show that $r + s$ colors is necessary to color the paired graph:
Let there be a k-coloring of the paired graph. Then the colors used for vertices of G must be disjoint from the colors used for vertices of H, since there is an edge between every vertex of G and every vertex of H. Now, let p colors be used for vertices in G and $k - p$ colors be used for vertices H. Then p colors are sufficient for coloring the graph G and hence $p \geq r$. And since $k - p$ colors are sufficient for coloring the graph H, $k - p \geq s$. Hence $k \geq r + s$.

4. **[15 points]**

For any graph $G = (V, E)$, let us say that G *has a full connected 2-colorable subgraph* if there is a subgraph $H = (V, E')$ of G (containing all vertices of G) that is 2-colorable and connected.

Note that in the above definition, we require H to include *all* vertices of G.

Prove that every connected graph G has a full connected 2-colorable subgraph.

You are required to give an elementary proof that only assumes the definitions of colorability. You cannot assume properties of graphs that you may know.

Hint: Choosing the right variable to induct on can make the proof easier.

Solution:

We will prove the claim by induction on the number of edges.
We will prove that $P(m)$ holds for every $m \in \mathbb{N}$, where

$$P(m): \textit{ every connected graph } G \textit{ with } m \textit{ edges has a full connected 2-colorable subgraph.}$$

Base case: $m = 0$:
A connected graph G with no edges can have only one vertex. The graph G is a subgraph of itself, and is clearly 2-colorable and connected and has all vertices of G. Hence G has a full connected 2-colorable subgraph.

Induction step:

Let $m > 0$ and let G be an arbitrary connected graph with n edges.

We assume the induction hypothesis:

> Any connected graph with k edges, where $k < m$, has a full connected 2-colorable subgraph.

Since G has at least one edge, let us pick an edge (u, v) in G.
Let G' be the graph obtained by removing this edge.

Let us consider two cases.

Case 1: G' is connected
If G' is connected, then by the induction hypothesis, since G' has less than m edges, there is a full connected 2-colorable subgraph H of G'.
This subgraph is also clearly a full connected 2-colorable subgraph of G (it has all vertices of G, it is 2-colorable, and it is connected).

Case 1: G' is not connected
Since G was connected and G' is obtained by removing an edge of G, G' consists of two connected components: a connected component R consisting of the vertices connected to u and a connected component S consisting of the vertices connected to v.

Since R and S are connected graphs and have less than m edges, by the induction hypothesis, there is a full connected subgraph H_1 of R and a full connected subgraph H_2 of S.

Now consider the graph H obtained by taking all vertices of H_1 and H_2 and the union of edges in H_1 and H_2 and the edge (u, v). This graph clearly spans all vertices of G.

It is also 2-colorable: let us take a 2-coloring of H_1 with colors red and blue, and a 2-coloring of H_2 with red and blue. If u and v are colored differently, then this gives a two-coloring of H. If not, we can recolor H_2 by swapping red and blue so that it remains a 2-coloring, and u and v are colored differently, giving again a 2-coloring of H.

Also, since H_1 is connected and H_2 is connected, and the edge (u, v) connects a vertex of H_1 with a vertex of H_2, H is connected.

Hence G has a full connected 2-colorable subgraph. Q.E.D.

5. **[15 points]**

Professor Graphnütter makes the following claim:

For any $n \in \mathbb{N}$, $n > 0$, and for any connected graph G with $2n$ vertices,
there exists n different edges $e_1, \ldots e_n$ of G such that they do not share a common vertex
(i.e., no two edges are incident on the same vertex).

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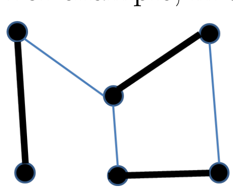
For example, in the following graph, the edges marked bold is a set of edges that satisfy the claim.



Professor Graphnütter also gives the intuition as to why this should be true: *"You should be able to find an edge between some pair of vertices, and remove them, and continue this way, and thus find n such edges."*

Either prove the claim to be correct (using induction), or disprove the claim.

Solution:

Professor Graphnütter is wrong. The following graph has 4 vertices and is connected, but there are no pair of edges that do not share a common vertex.