CS 173, Fall 2014
Final Exam (comprehensive)

NAME:

NETID (e.g., aturing24):

Circle your discussion:

Fri-11am **Fri-12am** **Fri-1pm** **Fri-2pm** **Fri-3pm** **Fri-4pm** **PROFIC**

Problem	1	2	3	4	5	Total
Possible	18	15	32	20	15	100
Score						

The exam is designed for 120 minutes. However, you have 180 minutes to finish the exam.

We will be checking photo ID's during the exam. Have your ID handy. Also put out your row number token.

Turn in your exam at the front when you are done.

INSTRUCTIONS (read carefully)

- There are 5 problems spanning 7 sheets of paper (including this cover sheet). Make sure you have a complete exam. The point value of each problem is indicated next to the problem, and on the cover page table. The last two pages are for scratch work.
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- When writing proofs, use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order.
- Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. Partial credit for multiple-choice questions is very rare.
- It is not necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, 3^{17}, and $7!$, unless it is explicitly indicated to completely simplify.
- It is wise to skim all problems and point values first, to best plan your time.
- This is a closed book exam.
  Turn off your cell phone now.
  No notes or electronic devices of any kind are allowed.
  These should be secured in your bag and out of reach during the exam.
- Do all work in the space provided, using the backs of sheets if necessary.
  If your work is on the backside then you must clearly indicate so on the problem.
  See the proctor if you need more paper.
- Please bring any apparent bugs or ambiguity to the attention of the proctors.
- After the exam is over, you may discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a makeup exam).

Problem 1: Multiple choice (3+3+3+3+3+3 = 18 points)

In this question, there are some problems where *muliple* choices could be marked, and some where a *single* choice needs to be marked. Read the questions carefully.

(a) A function $f : \mathbb{Z} \to \mathbb{N}$ is said to be grumpy if for every $y \in \mathbb{N}$, $y \neq 0$, there are precisely two distinct elements, $a, a' \in \mathbb{Z}$ such that $f(a) = f(a') = y$. Which of the following is a grumpy function?

Check all that apply.

- $f(x) = x$ ☐
- $f(x) = x^2$ ☐
- $f(x) = x^3$ ☐
- $f(x) = |x|$ ☐
- $f(x) = 0$ if x is even, and 1 otherwise ☐
- $f(x) = 0$ ☐

(b) Which of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ are injective (i.e., one-to-one) functions?
Check all that apply.

- $f(x) = x$ ☐
- $f(x) = x^2$ ☐
- $f(x) = x^3$ ☐
- $f(x) = |x|$ ☐
- $f(x) = 0$ if x is even, and 1 otherwise ☐
- $f(x) = 0$ ☐

(c) Which of the following are countably infinite sets?
Check all that apply.

- $\mathbb{N}$ ☐
- $\mathbb{R}$ ☐
- $\mathcal{P}(\mathbb{N})$ ☐
- Set of all C programs ☐
- Set of all functions from $\mathbb{N}$ to $\{0, 1\}$ ☐
- Set of all pairs of natural numbers ☐

(d) For any $a \in \mathbb{N}$ with $a > 1$, $(a^3 + 1) \equiv 1 \ mod \ a$ True ☐ False ☐

(e) Let S be a set with n elements. How many functions are there from $S \times S$ to S?

Choose *only* one option.

n ☐	n^2 ☐	n^3 ☐	2^n ☐
2^{n^2} ☐	$n^{(n^2)}$ ☐	n^{2n} ☐	$n!$ ☐

(f) The problem of checking whether a C program prints "Yes" on the screen is Decidable ☐ Undecidable ☐

Problem 2: Short problems (3+3+3+3+3=15 points)

(a) Give the negation of the following formula, phrasing it such that negations are only at the level of individual propositions and there are no implications (push negations in).

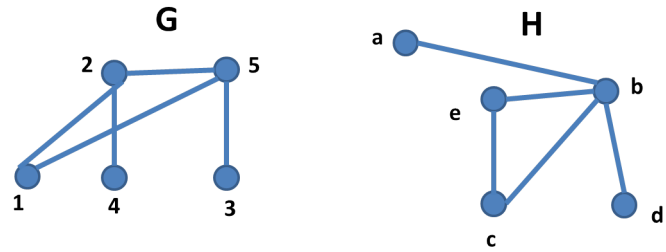
$$(p \vee \neg q) \Rightarrow (\neg r \wedge s)$$

(b) What is the contrapositive of the following statement?

For any real number, if it is not semi-algebraic, then it is not co-rational or it is quadratic.

(c) A wine store has many bottles of Syrah, Merlot, and Malbec. How many ways are there of choosing four wine bottles to buy from this store?

(d) Consider the following two graphs G and H:



Are they isomorphic? If yes, give a bijection between the vertices exhibiting the isomorphism; if not, give a simple argument as to why they are not isomorphic.

(e) Let $T(1) = 1$
and $T(n) = 3T(n-1) + T(n-2) + c$, for $n > 1$.

Write $T(n)$ in terms of $T(n-2)$ and $T(n-3)$ only (for $n \geq 4$).

Problem 3: Simple proofs (8+8+8+8 = 32 points)

(a) For all real numbers p and q, if $p^2 q \leq 100$ then $p \leq 5$ or $q \leq 5$.

(b) Prove the following formally (informal answers will get no credit).
There is no *finite* set $S \subseteq \mathbb{N}$ such S contains at least one non-zero element, and S satisfies the following property:

$$\forall x, y \in S. \quad (x + y) \in S$$

(c) In this question, $\mathcal{P}(\mathcal{S})$ denotes the power-set of S.

Let A and B be two sets. Prove that if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{B})$, then $A \subseteq B$.

(d) A relation R on a set A is said to be a pre-order if R is reflexive and transitive.

Let G be an undirected graph with vertex set V.
Let R_G be the relation on V given by:

$$u \; R_G \; v \quad \textit{iff} \quad v \quad \textit{is reachable from} \quad u.$$

Prove that R_G is a preorder.

(Note: In a graph, a vertex v is reachable from a vertex u if there is a path from u to v.)

Problem 4: Induction on Trees (5+15= 20 points)

Consider rooted ternary (3-ary) trees (trees where every node has 0, 1, 2, or 3 children).
A *frivolous* tree is a rooted ternary tree T which has a natural number labeling every vertex, such that:

- Each leaf is labeled with either 3 or 5.
- No node has exactly one child.
- If a node u has two children, and they are labeled r and r', then the label of u is rr'.
- If a node u has three children with labels r, r' and r'', then the label of u is $r + r' + r''$.

a) Draw a frivolous tree with at least 7 nodes, and that has at least one node with two children, and that has at least one node with three children. Mark the labels of the nodes clearly.

b) Prove by induction that in any frivolous tree T the label of the root of T is odd.

Problem 5: Proofs on Graphs (5+10= 15 points)

For every $n \geq 1$, the *ladder* graph of rank n is defined to be the graph G_n with vertices

$$V_n = \{p_i \mid 1 \leq i \leq n\} \cup \{q_i \mid 1 \leq i \leq n\}$$

and with precisely the following edges:
for every $0 \leq i \leq n$, (p_i, q_i) is an edge, and for every $0 \leq i < n$, (p_i, p_{i+1}) is an edge and (q_i, q_{i+1}) is an edge.

a) Draw the ladder graph of rank 4. Label the vertices clearly.

b) Prove (by induction or in any other formal way) that for every n, the ladder graph of rank n is 2-colorable.