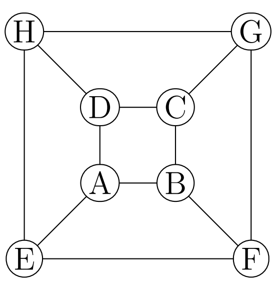
CS 173 Section B, Spring 2016, Examlet 3

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

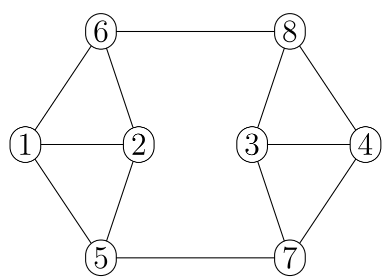
Problem	1	2	3	4	5	6	7	Total
Possible	5	15	10	6	14	10	10	70
Score								

1. [**5 points**]

Are the graphs shown below isomorphic? Justify your answer.



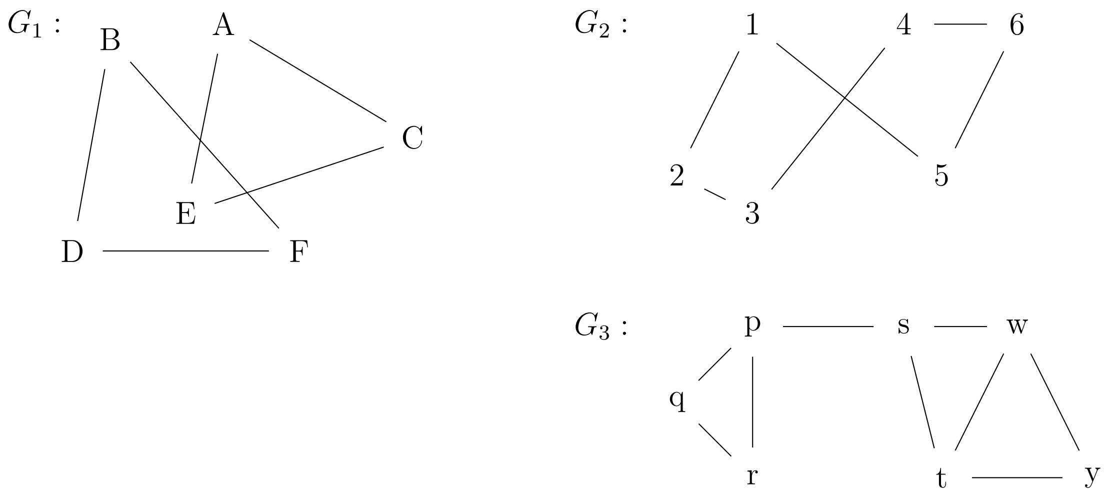
H_1



H_2

Solution:
The graphs are not isomorphic.
To see this, notice that H_2 has a 3-clique as a subgraph $(1, 2, 5)$, while there is no 3-clique subgraph of H_1.

2. **[5+5+5=15 points]**



Consider the three graphs G_1, G_2 and G_3 depicted above.

(a) How many connected components does each graph have?

Solution:
G_1: 2
G_2: 1
G_3: 1

(b) What is the diameter of each of the graphs?

Solution:
G_1: ∞ (since the graph is not connected)
G_2: 3 (one of the shortest paths of length 3 is $1 \to 5 \to 6 \to 4$)
G_3: 4 (one of the shortest paths of length 4 is $r \to p \to s \to t \to y$)

(c) What is the minimum number of colors you need to color each of the graphs? Justify your answer.

Solution:

G_1: 3 colors
A 3-coloring: $A \to$ *red*, $E \to$ *blue*, $C \to$ *green*, $B \to$ *red*, $D \to$ *blue*, $F \to$ *green*.
Moreover, 3 colors are essential as the graph has a 3-clique (e.g., A, E, C).

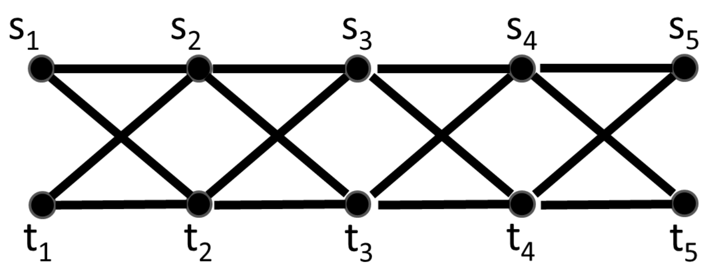
G_2: 2 colors
A 2-coloring: $1 \to$ *red*, $2 \to$ *blue*, $3 \to$ *red*, $4 \to$ *blue*, $6 \to$ *red*, $5 \to$ *blue*.
Moreover, 2 colors are essential as the graph has an edge.

G_3: 3 colors
A 3-coloring: $p \to$ *red*, $q \to$ *blue*, $r \to$ *green*, $s \to$ *blue*, $t \to$ *red*, $w \to$ *green*, $y \to$ *blue*.
Moreover, 3 colors are essential as the graph has a 3-clique (e.g., p, q, r).

3. **[4+6=10 points]**
For every $n \in \mathbb{N}$, $n > 0$, let $G_n = (V_n, E_n)$ be the graph where

$$\begin{aligned} V_n &= \{s_1, s_2, \ldots, s_n\} \cup \{t_1, t_2, \ldots, t_n\} \\ E_n &= \{s_i t_{i+1} \mid 1 \leq i < n\} \\ &\cup \{s_{i+1} t_i \mid 1 \leq i < n\} \\ &\cup \{s_i s_{i+1} \mid 1 \leq i < n\} \\ &\cup \{t_i t_{i+1} \mid 1 \leq i < n\} \end{aligned}$$

(a) Draw the graph G_5.



(b) Prove that for any $n > 0$, G_n is 2-colorable.
You should give a precise assignment of colors to nodes of G_n, for every n, that gives a valid 2-coloring.

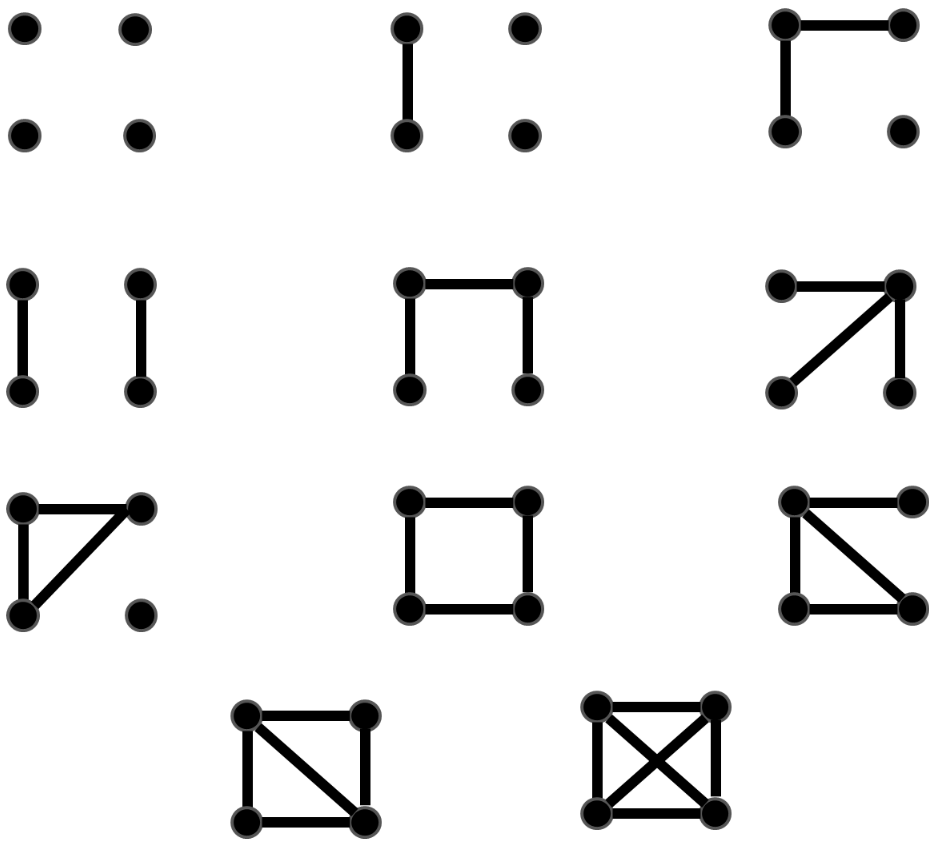
Let $n > 0$. Let us color G_n as follows.
Assign every node s_i and t_i the color *red*, when i is odd, and *blue*, when i is even.
Clearly this is a 2-coloring.
For the edges of the kind $s_i t_{i+1}$, the two vertices have different color.
For the edges of the kind $s_{i+1} t_i$, the two vertices have different color.
For the edges of the kind $s_i s_{i+1}$, the two vertices have different color.
For the edges of the kind $t_i t_{i+1}$, the two vertices have different color.

4. (Enumerate graphs) **[6 points]**

Draw *all* graphs with 4 vertices, without repeating isomorphic graphs.

You do not have to label the vertices.

Solution:



5. **[14 points]**

Mark True or False for each of the questions below.
Think carefully. Multiple choice questions are not necessarily easy!

(a) Let $f : A \to B$ and $g : B \to A$. If $g \circ f : A \to A$ is one-to-one, then f is one-to-one.

True [X] False [ ]

(b) Let $f : A \to B$ and $g : B \to A$. If $g \circ f : A \to A$ is onto, then f is onto.

True [ ] False [X]

(c) Let $f : A \to B$ and $g : B \to A$. If $g \circ f : A \to A$ is one-to-one, then g is one-to-one.

True [ ] False [X]

(d) Let $f : A \to B$ and $g : B \to A$. If f is one-to-one and g is onto, then $g \circ f : A \to A$ is onto.

True [ ] False [X]

(e) Let $f : A \to B$ and $g : B \to A$, where A and B are finite sets. If f is one-to-one and g is one-to-one, then $|A| = |B|$.

True [X] False [ ]

(f) Let $f : A \to B$ and $g : B \to A$. If $g \circ f : A \to A$ is the identity function (i.e., $g \circ f(x) = x$, for every $x \in A$), then f is one-to-one.

True [X] False [ ]

(g) Let $f : A \to B$ and $g : B \to A$. If $g \circ f : A \to A$ is the identity function (i.e., $g \circ f(x) = x$, for every $x \in A$), then f is onto.

True [ ] False [X]

6. **[10 points]**

Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is an onto function.
Let us define $g : (\mathbb{Z} \times \mathbb{Z}) \to (\mathbb{Z} \times \mathbb{Z})$ as

$$g(x, y) = (f(x) + y, y + 3), \qquad \textit{for every } x, y \in \mathbb{Z}$$

Prove that g is onto.

Solution:
Let $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ be an arbitrary pair of integers.

Since $p - q + 3$ is an integer and f is onto, we know that there is an integer $x \in \mathbb{Z}$ such that $f(x) = p - q + 3$.
Let $y = q - 3$; clearly $y \in \mathbb{Z}$
Now observe that $g(x, y) = g(x, q - 3) = (f(x) + q - 3, q - 3 + 3) = (p - q + 3 + q - 3, q) = (p, q)$

Hence for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, there exists $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(x, y) = (p, q)$.
Hence g is onto. **Q.E.D.**

7. **[10 points]**

Let P be the set of positive natural numbers.
Let $f : P \times P \to \mathbb{R}^2$ be the function that maps $f(x, y) = (\frac{2x}{y}, x + y + 1)$, for every $x, y \in P$.

Prove that f is a one-to-one function. You must work directly from the definition of one-to-one. Do not use any other facts (for example, about the behavior of increasing functions) that you may know.

Solution:

Let $(x, y), (p, q) \in P \times P$ and assume $f(x, y) = f(p, q)$.
Then $(\frac{2x}{y}, x + y + 1) = (\frac{2p}{q}, p + q + 1)$.
Hence $\frac{2x}{y} = \frac{2p}{q}$ and $x + y + 1 = p + q + 1$.
Hence $xq = py$ and $x + y = p + q$.

Hence $x = \frac{py}{q}$.
Substituting x in $x + y = p + q$ we get $\frac{py}{q} + y = p + q$.
I.e., $y(p + q) = q(p + q)$.
Since $p, q \in P$, $p + q \neq 0$.
Hence $y = q$.

Substituting $y = q$ in $x + y = p + q$, we get $x + q = p + q$, i.e., $x = p$.

Hence $(x, y) = (p, q)$.
Hence f is one-to-one. **Q.E.D.**