CS 173, Spring 2016, Examlet 2, Part A

LASTNAME, FIRSTNAME (in CAP letters):   **NETID:**

Problem	1	2	3	4	5	6	Total
Possible	5	15	5	15	5	15	60
Score							

1. **[5 points]**
   We say a relation R over $\mathbb{N}$ is *zero-liking* if it is an equivalence relation, and moreover for every $i \in \mathbb{N}, \quad i\ R\ 0$.

   Which of the following are true about zero-liking relations?

	True	False
There are no zero-liking relations		X
There is precisely one zero-liking relation	X	
There are more than one zero-liking relations		X
For any zero-liking relation R and $i, j \in \mathbb{N}$, iRj	X	
For any zero-liking relation R and $i, j \in \mathbb{N}$ and $i > 0, j > 0$, $i \not R j$		X

2. **[15 points]**
   Let S be an arbitrary nonempty set and let R be an equivalence relation on S.
   Let T be the relation:

   $$aTb \text{ iff } \neg(aRb), \forall a, b \in S$$

   In other words, two elements are related by T iff they are not related by R.

   In the following, you are either asked to prove a property of T or give a counterexample. A counterexample is a concrete set S and a concrete relation R on S such that the corresponding relation T does not have the specified property.

a) Is T always irreflexive? If yes, give a proof. If no, give a counterexample.

Solution: Yes, T is always irreflexive.

Proof: Let $a \in S$ be an arbitrary element. Since R is an equivalence relation, it is reflexive. Hence aRa.Hence $a \not T a$. Hence T is irreflexive. ________ **QED.**

b) Is T always symmetric? If yes, give a proof. If no, give a counterexample.

Solution: Yes, T is always symmetric.

Proof: Let $a, b \in S$ be arbitrary elements and assume aTb. Then $a \not R b$.
Hence $b \not R a$ (since R is an equivalence relation and hence is symmetric).
Hence bTa.
Hence T is symmetric. ________ **QED.**

c) Is T always transitive? If yes, give a proof. If no, give a counterexample.

Solution: No, T need not be transitive.

Counterexample: Let $S = \{a, b, c\}$ and let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$.
Clearly, R is an equivalence relation.
Then $T = \{(a, c), (c, a), (b, c), (c, b)\}$.
Note that T is not transitive, as aTc and cTb, but $a \not T b$.

CS 173, Spring 2016, Examlet 2, Part B

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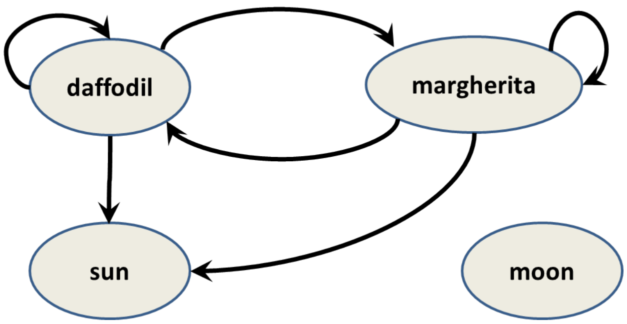
3. [**5 points**]
Let the set A be $A = \{daffodil, margherita, sun, moon\}$, Give a relation R over A that is not symmetric, that is not antisymmetric, but is transitive. Give the relation either as a set of pairs or a directed graph.



4. (Induction) [**15 points**]

Prove the following, for every natural number $n > 0$

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Note that

$$\sum_{i=1}^{n} i(i+1)(i+2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2)$$

.

Proof:
We will prove by induction on n, that for every $n \in \mathbb{N}$ with $n > 0$,

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

.

Base case:
When $n = 1$,

$$\sum_{i=1}^{n} i(i+1)(i+2) = 1 \cdot 2 \cdot 3 = 6 \ \textit{and} \ \frac{n(n+1)(n+2)(n+3)}{4} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} = 6$$

.
Hence the claim holds when $n = 0$.

Induction step: Let $k > 1$ be an arbitrary natural number.

Let us assume the **induction hypothesis**: For every $1 \leq j < k$,

$$\sum_{i=1}^{j} i(i+1)(i+2) = \frac{j(j+1)(j+2)(j+3)}{4}$$

$$\begin{aligned}
\sum_{i=1}^{k} i(i+1)(i+2) &= \sum_{i=1}^{k-1} i(i+1)(i+2) \quad + \quad k(k+1)(k+2) \\
&= \frac{(k-1)k(k+1)(k+2)}{4} + k(k+1)(k+2) \qquad \textit{(by the induction hypothesis)} \\
&= \frac{(k-1)k(k+1)(k+2) + 4k(k+1)(k+2)}{4} \\
&= \frac{k(k+1)(k+2)(k-1+4)}{4} \\
&= \frac{k(k+1)(k+2)(k+3)}{4}
\end{aligned}$$

Hence we have proved the claim by induction.________________________________ **QED**

CS 173, Spring 2016, Examlet 2, Part C

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5. [**5 points**]

I want to prove that for every natural number $n \in \mathbb{N}$ with $n > 2$, a statement $P(n)$ holds.

Which of the following ways are valid ways of proving the above.

Check all that apply.

I prove $P(3)$ holds and prove $P(4)$ holds and prove that for every $k > 4$, if $P(k-2)$ holds then $P(k)$ holds.	X
I prove that $P(3)$ holds and prove that for every $k > 3$, if $P(i)$ does not hold for some $i < k$ then $P(k)$ does not hold.	
I prove that $P(3)$ holds and prove that for every $k > 3$, if $P(k)$ does not hold, then $P(i)$ does not hold, for some $3 \leq i < k$.	X
I prove that $P(3)$ holds and prove that for every $k > 3$, if $P(k-2)$ holds then $P(k)$ holds.	
I prove that $P(3)$ holds, $P(4)$ holds, $P(5)$ holds, prove that for every $k > 5$, if $P(k)$ does not hold, then $P(k-2)$ does not hold.	X

6. Induction [**15 points**]

Prove that for every $n \in \mathbb{N}$ with $n \geq 18$, there exists $i, j \in \mathbb{N}$ such that $2^n = 8^i \cdot 1024^j$.

(Note: $8 = 2^3$ and $1024 = 2^{10}$).

Solution:

We will prove by induction on n that for every $n \geq 18$, there exists $i, j \in \mathbb{N}$ such that $2^n = 8^i \cdot 1024^j$.

<u>Base cases:</u>
When $n = 18$, $2^{18} = 8^6 \cdot 1024^0$.
When $n = 19$, $2^{19} = 8^3 \cdot 1024^1$.
When $n = 20$, $2^{20} = 8^0 \cdot 1024^2$.

<u>Induction step:</u> Let $k > 20$ be an arbitrary natural number.

Let us assume the **induction hypothesis**:
For every $18 \leq r < k$, there exists $i, j \in \mathbb{N}$ such that $2^r = 8^i \cdot 1024^j$.

Since $18 \leq k - 3 < k$, by the induction hypothesis, we know that there exists $i, j \in \mathbb{N}$ such that $2^{k-3} = 8^i \cdot 1024^j$.

Hence $2^k = 2^3 \cdot 2^{k-3} = 8 \cdot 8^i \cdot 1024^j = 8^{i+1} \cdot 1024^j$.

Since $i + 1, j \in \mathbb{N}$, we have shown that $2^k = 8^s \cdot 1024^t$ for some $s, t \in \mathbb{N}$.

Hence we have proved the claim by induction.____ **QED.**