Homework 6

Discrete Structures
CS 173 [B] : Fall 2015

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Due: Fri Apr 17, 5:00 PM

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PART 1 (Machine-Graded Problems) on Moodle. [25 points]

PART 2 [75 points]

1. **Recurrence Relation** [20 points]

   Recall that $\binom{n}{k}$ is the number of subsets of size k that a set of size n has.

   (a) Use mathematical induction to prove that, for all $n, k \in \mathbb{N}$ such that $k \leq n$, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, based on the following: $\forall n \in \mathbb{N}$, $\binom{n}{0} = \binom{n}{n} = 1$; and, for $n \geq 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (which we obtained by considering separately the subsets of size k that contain and do not contain a fixed element from the set).

   Solution: We need to prove that $\forall n \in \mathbb{N}$ the following holds: $\forall k \in \mathbb{N}, k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   We prove this statement by induction on n.
   The base case: Consider $n = 0$. Then, the only value of $k \in \mathbb{N}$, $k \leq n$ is $k = 0$. For $n = 0, k = 0$, we have $\binom{n}{k=1}$ (by a base case of the recursive definition) and $\frac{n!}{k!(n-k)!} = \frac{1}{1 \cdot 1} = 1$.
   Induction step:
   Suppose that for some $n_0 \in \mathbb{N}$, for all $n \leq n_0$, it holds that $\forall k \in \mathbb{N}, k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   Then we shall prove that, for all $k \in \{0, \ldots, n_0 + 1\}$, $\binom{n_0+1}{k} = \frac{(n_0+1)!}{k!(n_0+1-k)!}$.
   First, for $k = 0$ and $k = n_0 + 1$ this is given by the base cases of the recurrence relation.
   Now consider $k \in \{1, \ldots, n_0\}$. Since $n_0 + 1 \geq 1$, we can apply the recurrence relation to obtain

$$\begin{aligned}
\binom{n_0+1}{k} &= \binom{n_0}{k-1} + \binom{n_0}{k} \\
&= \frac{n_0!}{(k-1)!(n_0-k+1)!} + \frac{n_0!}{k!(n_0-k)!} && \text{by IH, since } 0 \leq k-1, k \leq n_0 \\
&= \frac{n_0!}{(k-1)!(n_0-k)!}\left(\frac{1}{n_0-k+1} + \frac{1}{k}\right) \\
&= \frac{n_0!}{(k-1)!(n_0-k)!}\left(\frac{n_0+1}{n_0-k+1 \cdot k}\right) \\
&= \frac{(n_0+1) \cdot n_0!}{(k \cdot (k-1)!) \cdot ((n_0-k+1) \cdot (n_0-k)!)} \\
&= \frac{(n_0+1)!}{k! \cdot (n_0-k+1)!}
\end{aligned}$$

Note that the induction hypothesis could be applied to rewrite both $\binom{n_0}{k-1}$ and $\binom{n_0}{k}$, since $0 \leq k-1 \leq n_0$ and $0 \leq k \leq n_0$ (because $k \in \{1, \ldots, n_0\}$).

Thus, we have shown that if the induction hypothesis holds, then for all $k \in \{0, \ldots, n_0+1\}$, $\binom{n_0+1}{k} = \frac{(n_0+1)!}{k!(n_0+1-k)!}$.

By mathematical induction, this proves that for all $n \in \mathbb{N}$, for all $k \in \{0, \ldots, n\}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(b) Above, $\binom{n}{k}$ was expressed in terms of $\binom{n-1}{i}$ for two different values of i. Use a similar argument to express $\binom{n}{k}$ in terms of $\binom{n-2}{i}$ for different values of i (for $n \geq 2$).

[Hint: Alternately, note that $\binom{n}{k}$ is the coefficient of x^k in the expansion of $(1+x)^n = (1+x)^2 \cdot (1+x)^{n-2}$.]

Solution: To choose a subset of k elements from a set S with two elements a, b and $|S| = n$, one has four options:

- Include both a and b in the subset: the remaining elements can be chosen in $\binom{n-2}{k-2}$ ways.
- Include a but not b in the subset: the remaining elements can be chosen in $\binom{n-2}{k-1}$ ways.
- Include b but not a in the subset: the remaining elements can be chosen in $\binom{n-2}{k-1}$ ways.
- Include neither a nor b in the subset: the remaining elements can be chosen in $\binom{n-2}{k}$ ways.

Summing up, there are $\binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ ways of choosing a subset of size k from a set of size n (with $n \geq 2$).

2. **Partitions from Onto Functions.** [20 points]

Consider the following definitions.

- For a function $f : A \to B$, let $\hat{f} : A \to \text{Image}(f)$ be the unique *onto function* such that $\forall x \in A$ $f(x) = \hat{f}(x)$.
- For a function $g : A \to C$, let the pre-image function $PI_g : C \to \mathbb{P}(A)$ be defined by $PI_g(y) = \{x \mid f(x) = y\}$.
- For a function $f : A \to B$, let the "pre-image partition" of A, be defined as $PP_f = \text{Image}(PI_{\hat{f}})$.
- Define an equivalence relation $\sim$ between functions $f_1 : A \to B$ and $f_2 : A \to B$ as follows: $f_1 \sim f_2$ if $PP_{f_1} = PP_{f_2}$.

Answer the following with respect to the above definitions.

(a) Suppose $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Consider $f : A \to B$ defined as $f(a) = f(b) = 1$ and $f(c) = 2$. Also, let $f' : A \to B$ be defined as $f'(a) = f'(b) = 3$ and $f'(c) = 2$

i. Describe the functions $\hat{f}$ and $\hat{f}'$.

Solution: $\hat{f} : A \to \{1, 2\}$ is defined as $\hat{f}(a) = \hat{f}(b) = 1$ and $\hat{f}(c) = 2$. $\hat{f}' : A \to \{2, 3\}$ is defined as $\hat{f}'(a) = \hat{f}'(b) = 3$ and $\hat{f}'(c) = 2$.

ii. Describe the functions $PI_{\hat{f}}$ and $PI_{\hat{f}'}$.

Solution: $PI_{\hat{f}} : \{1, 2\} \to \mathbb{P}(A)$ is defined as $PI_{\hat{f}}(1) = \{a, b\}$, $PI_{\hat{f}}(2) = \{c\}$. $PI_{\hat{f}'} : \{2, 3\} \to \mathbb{P}(A)$ is defined as $PI_{\hat{f}}(2) = \{c\}$, $PI_{\hat{f}}(3) = \{a, b\}$.

iii. Describe the partitions PP_f and $PP_{f'}$.

Solution: $PP_f = PP_{f'} = \{\{a, b, \}, \{c\}\}$.

(b) Let $f : A \to B$, where $|A| = n$, $|B| = k$ and $|\text{Image}(f)| = i$. Then how many functions f' are there such that $f \sim f'$? Justify your answer.

Solution: PP_f consists of i non-empty sets, each of which has all its members mapped to the same value in B. That is, f labels each of the i sets in PP_f with a distinct element in B. A function f' such that $PP_f = PP_{f'}$ can be chosen by choosing any i distinct values as labels for the i sets in PP_f. Since B has k elements, this can be done in $P(k, i)$ ways.

Thus there are $P(k, i) = \frac{k!}{(k-i)!}$ functions f' such that $f \sim f'$.

3. **Lottery** [20 points]

Counting is intimately connected to computing the *probability* of various events. In this problem we shall use counting to calculate the probability of winning lotteries.

In a certain kind of lottery, each player submits a sequence of n digits (between 0 and 9). A player wins a grand prize if her submission exactly matches a sequence of n digits selected by a random mechanical process. She wins a smaller prize if only $n - 1$ digits are matched (e.g., for $n = 4$, if the submission is 1248 but the machine chooses 1298, then a small prize is awarded).

(a) How many ways can the mechanical process choose a sequence of n digits? Use this to compute the probability of a player (who has submitted a single sequence) winning the large prize, assuming that the mechanical process chooses each possible sequence equally likely (i.e., uniformly at random).

[Hint: You can use the following fact regarding probability. If one item is chosen out of N possible items uniformly at random, then the probability of it being any priori fixed item is $1/N$.]

Solution: A sequence of n digits can be chosen in $N = 10^n$ ways. Hence, the probability that this matches the one submitted by a given player is $1/10^n$.

(b) For any sequence of n digits that a player picks, how many sequences are there which, if chosen by the mechanical process, would result in the player winning a small prize? Use this to compute the probability that a player (who has submitted a single sequence) wins the small prize.

[Hint: The probability in this case is $\frac{p}{N}$, where p is the number of sequences, which if chosen by the mechanical process, leads to a small prize, and N is the total number of all possible sequences that the mechanical process can choose.]

Solution: To choose a sequence that differs in exactly one position from a given n digit number, we can first choose the position where it differs (n ways), and then choose a digit for that position which is different from the original digit (9 ways). Thus there are $p = 9n$ strings which, if chosen by the mechanical process, will yield a smaller prize. In all, there are $N = 10^n$ strings. The probability of getting a smaller prize is therefore $\frac{9n}{10^n}$.

4. **Sorted Strings** [15 points]

Consider strings made up of lowercase letters, `a-z`. We say that a string is a "sorted string" if the letters in it appear in alphabetic order. For instance, `bbn` and `tux` are sorted strings, but `ibm` is not.

(a) How many sorted strings of length 3 are there?

[Hint: Can you relate a sorted string to a multi-set?]

Solution: There is a bijection between the set of all sorted strings of length 3 and the set of all multi-sets of size 3 (with elements from the set of all letters).

Hence the number of sorted strings of length 3 is equal to the number of size-3 multi-sets of letters. This is equal to the number of ways in which 3 balls can be thrown into 26 bins. By the "stars-and-bars" technique, this is equal to $\binom{28}{3}$.

(b) How many sorted strings of length 3 are there in which no letter repeats? (Thus `bbn` should not be counted, but `tux` should be.)

Solution: There is a bijection between the set of all sorted strings of length 3 with no repetitions and the set of all *sets* of size 3 (with elements from the set of all letters).

Hence the number of sorted strings of length 3 is equal to the number of size-3 sets of letters. This is simply $\binom{26}{3}$.