Homework 5

Discrete Structures
CS 173 [B] : Fall 2015

Released: Tue Mar 24
Due: Fri Apr 3, 5:00 PM

Submit on Moodle.

PART 1 (Machine-Graded Problems) on Moodle. [30 points]

PART 2 [70 points]

1. **Strong induction.** [15 points]

   An $a \times b$ chocolate bar is a rectangular piece of chocolate consisting of ab square pieces of chocolate. Your job is to break this chocolate into the ab individual square pieces. At any point during this task, you will have one or more pieces of the chocolate bar; you can pick any piece and break it into two, along a vertical or horizontal line separating the square pieces. For instance, if you start with a 2×2 bar, you can first break it vertically to get two 2×1 bars; then each of them you can break once horizontally, to end up with all 4 individual squares. In this process you made 3 breaks in all (one vertical, two horizontal).

   Show that to completely break an $a \times b$ bar into individual squares, you need exactly $ab - 1$ breaks, no matter which breaks you make.

   [Hint: Induct on $n = ab$; use strong induction. Use the fact that a single break splits a piece of chocolate into two smaller pieces with the same total number of squares. The rectangular geometry is not really important.]

2. **Golden Ratio** [15 points]

   Define a function $g : \mathbb{N} \to \mathbb{R}$ recursively as follows:

   - $g(1) = 1$
   - $g(n+1) = 1 + \frac{1}{g(n)}$ for all integers $n \geq 1$

   Recall that the Fibonacci numbers are defined recursively as follows:

   - $F_0 = 0, F_1 = 1$
   - $F_n = F_{n-1} + F_{n-2}$ for each integer $n \geq 2$

   Use induction to prove that $g(n) = F_{n+1}/F_n$ for all $n \in \mathbb{Z}^+$.

   Comment: As n tends to infinity, $g(n)$ tends to the positive solution of the quadratic equation given by $x = 1 + \frac{1}{x}$. This number, $\frac{1+\sqrt{5}}{2} \approx 1.618$ is sometimes called the "golden ratio."

3. **Bit Strings without Consecutive Zeros** [20 points]

A *bit-string* is simply a finite sequence of zeroes and ones. For the purposes of this problem, strings will always have length ≥ 1, i.e. no zero-length strings.

Let A_n be the number of strings of length n that end in a 1, and have no two consecutive zeros. Let B_n be the number of strings of length n that end in a 0, and have no two consecutive zeros. Thus $A_1 = 1$ and $B_1 = 1$. $A_2 = 2$ (strings 01 and 11) and $B_2 = 1$ (string 10).

(a) List A_n and B_n for $n = 3, 4$.

(b) Give recursive definitions for A_n and B_n in terms of A_{n-1} and B_{n-1} (each one possibly using both of them).

(c) What is A_n and B_n in terms of the Fibonacci numbers?

(d) How many bit-strings of length n are there in which there are no two consecutive zeros?

4. **Context-Free Grammar.** [20 points]

Consider the following grammar G whose set of non-terminals is $N = \{S, A, B\}$, the set of terminals is $\Sigma = \{a, b\}$, starting symbol S_0 is S, and the set of production rules P is given by:

- $S \to ASA \mid SBS \mid \epsilon$
- $A \to aSa \mid aa$
- $B \to bbS \mid bb$

(a) Give two examples of strings of terminals generated by G, which have parse-trees of height two.

(b) Prove that any strings of terminals generated by G will always have even numbers of both a's and b's.

Hint: You should prove a *stronger* statement: any tree of height $h \geq 1$ generated by G, with any of the non-terminals as start symbol, with only terminals at the leaves, has an even number of a's and an even number of b's. To prove this statement, use induction on trees, with the height as the induction variable.