Homework 5

Discrete Structures
CS 173 [B] : Fall 2015

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Due: Fri Apr 3, 5:00 PM

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PART 1 (Machine-Graded Problems) on Moodle. [30 points]

PART 2 [70 points]

1. **Strong induction.** [15 points]

   An $a \times b$ chocolate bar is a rectangular piece of chocolate consisting of ab square pieces of chocolate. Your job is to break this chocolate into the ab individual square pieces. At any point during this task, you will have one or more pieces of the chocolate bar; you can pick any piece and break it into two, along a vertical or horizontal line separating the square pieces. For instance, if you start with a 2×2 bar, you can first break it vertically to get two 2×1 bars; then each of them you can break once horizontally, to end up with all 4 individual squares. In this process you made 3 breaks in all (one vertical, two horizontal).

   Show that to completely break an $a \times b$ bar into individual squares, you need exactly $ab - 1$ breaks, no matter which breaks you make.

   [Hint: Induct on $n = ab$; use strong induction. Use the fact that a single break splits a piece of chocolate into two smaller pieces with the same total number of squares. The rectangular geometry is not really important.]

   Solution:

   Base Case: Consider $n = 1$. If we have 1 chocolate square, number of breaks $= 0$

   Inductive Hypothesis: For some $k \geq 1$, assume that claim is true for all chocolate bars with $n \leq k$ squares: that is, no matter what sequence of breaks are made, any such chocolate bar needs exactly $k - 1$ breaks.

   Induction Step: We need to show that claim is true for all chocolate bars with $n = k + 1$ squares.

   Consider an arbitrary chocolate bar with $k + 1$ squares. Consider any sequence of breaks that break it into individual squares. The first break breaks the bar into two pieces of n_1 and n_2 squares for some integers $n_1, n_2 > 0$ such that $n_1 + n_2 = k + 1$. Thus $n_1 \leq k$ and $n_2 \leq k$.

   Now, the sequence of breaks we are considering must break these two pieces into individual squares. Further, any break affects (pieces obtained from) only one of these two pieces. Thus the subsequent breaks can be partitioned into two sequences, one which breaks the n_1-square piece completely, and one which breaks the n_2-square piece completely. By the induction hypothesis, the first sequence has exactly $n_1 - 1$ breaks and the second sequence has exactly $n_2 - 1$ breaks.

So the total number of breaks in the original sequence (including the first break) is $1+(n_1-1)+(n_2-1) = n_1+n_2-1 = k$.

Thus we have shown that for any chocolate bar with $k+1$ squares, any sequence of breaks that break it into individual bars must have exactly k breaks. This completes the induction step.

2. **Golden Ratio** [15 points]

Define a function $g : \mathbb{N} \to \mathbb{R}$ recursively as follows:

- $g(1) = 1$
- $g(n+1) = 1 + \frac{1}{g(n)}$ for all integers $n \geq 1$

Recall that the Fibonacci numbers are defined recursively as follows:

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}$ for each integer $n \geq 2$

Use induction to prove that $g(n) = F_{n+1}/F_n$ for all $n \in \mathbb{Z}^+$.

Comment: As n tends to infinity, $g(n)$ tends to the positive solution of the quadratic equation given by $x = 1 + \frac{1}{x}$. This number, $\frac{1+\sqrt{5}}{2} \approx 1.618$ is sometimes called the "golden ratio."

Solution:

Claim: $g(n) = F_{n+1}/F_n$ for all $n \in \mathbb{Z}^+$.

Proof: We will induct on n.

Base case: When $n = 1$, we see that $g(1) = 1$ and $F_2/F_1 = (F_1 + F_0)/F_1 = (1+0)/1 = 1$.

Induction Hypothesis: Suppose $g(n) = F_{n+1}/F_n$ for each $n \in \{1, \dots, k\}$ for some $k \in \mathbb{Z}^+$.

Inductive Step: We shall show that $g(k+1) = F_{k+2}/F_{k+1}$. Observe

$$\begin{aligned}
g(k+1) &= 1 + \frac{1}{g(k)} && \text{[Using the def in (ii)]} \\
&= 1 + \frac{1}{F_{k+1}/F_k} && \text{[Induction Hypothesis]} \\
&= 1 + \frac{F_k}{F_{k+1}} = \frac{F_{k+1}+F_k}{F_{k+1}} && \text{[Algebra]} \\
&= \frac{F_{k+2}}{F_{k+1}}. && \text{[Recursive definition of } F_{k+2}\text{]}
\end{aligned}$$

Hence, $g(k+1) = F_{k+2}/F_{k+1}$ as required.

Thus by induction we have shown that $g(n) = F_{n+1}/F_n$ for all $n \in \mathbb{Z}^+$. □

3. **Bit Strings without Consecutive Zeros** [20 points]

A *bit-string* is simply a finite sequence of zeros and ones. For the purposes of this problem, strings will always have length ≥ 1, i.e. no zero-length strings.

Let A_n be the number of strings of length n that end in a 1, and have no two consecutive zeros. Let B_n be the number of strings of length n that end in a 0, and have no two consecutive zeros. Thus $A_1 = 1$ and $B_1 = 1$. $A_2 = 2$ (strings 01 and 11) and $B_2 = 1$ (string 10).

(a) List A_n and B_n for $n = 3, 4$.

Solution: $A_3 = 3$, $B_3 = 2$; $A_4 = 5$, $B_4 = 3$. ($A_5 = 8$, $B_5 = 5$.)

(b) Give recursive definitions for A_n and B_n in terms of A_{n-1} and B_{n-1} (each one possibly using both of them).

Solution:

Let us call the strings having no consecutive zeros as "good strings."

The set of good strings of length n that end in 1 have a bijection with the set of good strings of length $n-1$ (that may end in 0 or 1): the bijection simply chops off the last bit from the string (and the inverse of this bijection adds 1 to the end of a string). Thus, the number of good strings of length n which end in 1 = Number of good strings of length $n-1$ = Number of good strings of length $n-1$ which end in 1 + Number of good strings of length $n-1$ which end in 0.

Hence, $A_n = A_{n-1} + B_{n-1}$

The set of good strings of length n that end in 0 have a bijection with the set of good strings of length $n-1$ that end in 1. This is because good strings that end in 0 must have the last but one position have a 1 (otherwise the last two bits will be 0s and the string wouldn't be good). Thus, the number of good strings of length n which end in 0 = Number of good strings of length $n-1$ which end in 1.

Hence, $B_n = A_{n-1}$.

(c) What is A_n and B_n in terms of the Fibonacci numbers?

Solution:

From the previous part, we know that $A_n = A_{n-1}+B_{n-1}$ and $B_n = A_{n-1}$. So, $A_n = A_{n-1}+A_{n-2}$ which is same recurrence relation as Fibonacci numbers. But, we need to check base cases too. $A_1 = 1$, $A_2 = 2$, $A_3 = 3$. Hence, $A_n = F_{n+1}$ where F_k denotes k^{th} Fibonacci number.

$B_n = A_{n-1} = F_n$.

(d) How many bit-strings of length n are there in which there are no two consecutive zeros?

Solution:

Number of strings with no consecutive zeros = Number of strings with no consecutive zeros and ends in 1 + Number of strings with no consecutive zeros and ends in 0.

Number of strings with no consecutive zeros = $A_n + B_n = F_n + F_{n+1} = F_{n+2}$.

4. **Context-Free Grammar.** [20 points]

Consider the following grammar G whose set of non-terminals is $N = \{S, A, B\}$, the set of terminals is $\Sigma = \{a, b\}$, starting symbol S_0 is S, and the set of production rules P is given by:

- $S \to ASA \mid SBS \mid \epsilon$
- $A \to aSa \mid aa$
- $B \to bbS \mid bb$

(a) Give two examples of strings of terminals generated by G, which have parse-trees of height two.

Solution:

aaaa and *bb*.

(b) Prove that any strings of terminals generated by G will always have even numbers of both a's and b's.

Hint: You should prove a *stronger* statement: any tree of height $h \geq 1$ generated by G, with any of the non-terminals as start symbol, with only terminals at the leaves, has an even number of a's and an even number of b's. To prove this statement, use induction on trees, with the height as the induction variable.

Solution:

We shall follow prove the claim in the hint by induction on the height of the parse trees. That is, we shall show that for all $h \geq 0$, in all strings generated by the given grammar using parse trees of height h, the number of a's and b's are even.

Base case: The only strings of terminals generated by parse trees of height 1, with a non-terminal S, A or B in the root are ϵ, aa and bb respectively, all of which have an even number of a's and an even number of b's. Hence the claim holds for $h = 1$.

Induction step: We shall show that for all $k \geq 1$, if it holds that all strings generated by parse trees of height $h \leq k$ have even number of a's and even number of b's, then all strings generated by parse trees of height $h = k + 1$ have even number of a's and b's.

So, suppose that for some $k \geq 1$, all strings generated by parse trees of height $h \leq k$ have even number of a's and of b's. Now consider an arbitrary string that is generated by a parse tree of height $k + 1$. Consider the first derivation of this parse tree. It must be one of four cases.

Case 1: $S \rightarrow ASA$. Then the string must have the form $\alpha\beta\gamma$, where α and γ are strings of terminals generated by a subtree of the given tree, rooted at a node labeled by A, and β is a string of terminals generated by a subtree rooted at a node labeled by S. Each of these sub-trees has height strictly lesser than that of the whole tree. Therefore, they can be generated by parse trees of height $\leq k$. By induction hypothesis α has $2p_1$ a's and $2q_1$ b's, β has $2p_2$ a's and $2q_2$ b's, and γ has $2p_3$ a's and $2q_3$ b's for some integers $p_1, q_1, p_2, q_2, p_3, q_3$.

(Note: Here is where the stronger claim is important. If we were just proving the original claim, then the induction hypothesis will not tell us anything about the parse trees rooted at a node labeled by A.)

Then the given string must have $2(p_1 + p_2 + p_3)$ a's and $2(q_1 + q_2 + q_3)$ b's which are also even.

Case 2: $S \rightarrow SBS$. Then the string must have the form $\alpha\beta\gamma$, where α and γ are strings of terminals generated by two subtrees of the given tree, both rooted at nodes labeled by S, β is generated by a subtree rooted at a node labeled by B. Now, exactly as in the above case, we can apply the induction hypothesis to α, β, γ and conclude that the given string has an even number of a's and an even number of b's.

Case 3: $A \rightarrow aSa$. In this case the string must be of the form $a\alpha a$, where α is generated by a parse tree of height k rooted at S, and hence by the induction hypothesis, it has $2p$ a's and $2q$ b's for some integers p, q. Hence the given string has $2(p + 1)$ a's and $2q$ b's, which are also even.

Case 4: $B \rightarrow bbS$. In this case the string must be of the form $bb\alpha$, where α is generated by a parse tree of height k rooted at S, and hence it has $2p$ a's and $2q$ b's for some integers p, q. Hence the given string has $2p$ a's and $2(q + 1)$ b's, which are also even.

Thus in all 4 cases, the given string must have an even number of a's and an even number of b's.

Hence by mathematical induction, any string generated starting from any non-terminal of this grammar has an even number of a's and even number of b's. In particular, all strings generated starting from S has this property.