Homework 4

Discrete Structures
CS 173 [B] : Fall 2015

Released: Fri Mar 6
Due: Fri Mar 13, 5:00 PM

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Note: Throughout this homework, a "graph" stands for a *simple* graph.

1. **Matching Number.** [8 points]

   We say that a simple graph H is a *matching* if no vertex in H has degree more than 1. For a simple graph G, we define its *matching number* to be the maximum number of edges in any subgraph of G which is a matching.

   For each of the following graphs, compute its matching number: C_5, K_5, W_5, $K_{4,5}$.

2. **Complement of a Graph** [8 points]

   We define the *complement of a graph* as a graph which has the same vertex set, but with exactly those edges that are absent from the original graph. Formally, if $G = (V, E)$, its complement $\overline{G} = (V, \overline{E})$, such that $\overline{E} = K_V \setminus E$ where $K_V = \{\{a, b\} | a \in V, b \in V, a \neq b\}$.

   Match each graph on the left with a description of its complement:

   (a) K_4
   (b) C_4
   (c) $K_{1,3}$
   (d) P_4 (a path with 4 nodes)

   (a) A graph with no edges.
   (b) A graph with a single edge.
   (c) A path with two edges.
   (d) A matching with two edges.
   (e) A graph isomorphic to the original one.
   (f) A complete graph.
   (g) A cyclic graph.

3. **What is Wrong With this Proof?** [4 points]

   Claim: If every vertex in a graph has degee at least 1, then the graph is connected.

   Proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has degree at least 1, then the graph is connected.

   Base case: There is only one graph with a single vertex and it has degree 0. Therefore, $P(1)$ is vacuously true.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$.

Consider an n-vertex graph G in which every vertex has degree at least 1. By the induction hypothesis, G is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to G to obtain an $(n+1)$-vertex graph H. Since x must have degree at least one, there is an edge from x to some other vertex; call it y. Since y is connected to every other node in the graph, x will be connected to every other node in the graph. QED

- ☐ A. The proof needs to consider base case $n = 2$.
- ☐ B. The proof needs to use strong induction.
- ☐ C. The proof should instead induct on the degree of each node.
- ☐ D. The proof only considers $(n+1)$ node graphs with minimum degree 1 from which deleting a vertex gives a graph with minimum degree 1.
- ☐ E. The proof only considers n node graphs with minimum degree 1 to which adding a vertex with non-zero degree gives a graph with minimum degree 1.
- ☐ F. This is a trick question. There is nothing wrong with the proof!

4. **Triangle-Free and Claw-Free Graphs.** [20 points]

   Recall that an *induced subgraph* of G is obtained by removing zero or more vertices of G as well as all the edges incident on the removed vertices. (No further edges can be removed.) Formally, $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{a, b\} \mid a \in V', b \in V', \{a, b\} \in E\}$.

   A graph G is said to be H*-free* if no induced subgraph of G is isomorphic to H. For example, $G = (V, E)$ is K_3-free (or triangle free) if and only if there are no three distinct vertices a, b, c in V such that $\{\{a, b\}, \{b, c\}, \{c, a\}\} \subseteq E$.

   Prove that the complement of a K_3-free graph is a $K_{1,3}$-free graph.[1]

   [Hint: Prove the contrapositive.]

5. **Regular Graph.** [20 points]

   For any integer $n \geq 3$ and any even integer d with $2 \leq d \leq n - 1$, show that there exists a d-regular graph with n nodes, by giving an explicit construction.

   For full credit, describe your graph as (V, E) where $V = \mathbb{Z}_n$ and E is formally defined using modular arithmetic. (You may find it convenient to use S_a to denote $\{1, \ldots, a\} \subseteq \mathbb{Z}_n$.)

   [Hint: What would you do for $d = 2$? Then consider adding additional edges for larger values of d.]

6. **Prove using Induction.** [20 points]

   Prove that for any positive integer n, for any triangle-free graph $G = (V, E)$ with $|V| = 2n$, it must be the case that $|E| \leq n^2$.

7. **Prove using Strong Induction.** [20 points]

[1] The graph $K_{1,3}$ is often called the "claw" graph. So this problem can be restated as asking you to prove that the complement of a triangle-free graph is a claw-free graph.

Prove that for any graph G and any two nodes a and b in G, if there is a walk from a to b, then there is a path from a to b.

[Hint: Induct on the length of the walk.]