Homework 4

Discrete Structures
CS 173 [B] : Fall 2015

March 11, 2015

Solutions.

Note: Throughout this homework, a "graph" stands for a *simple* graph.

1. **Matching Number.** [8 points]

   We say that a simple graph H is a *matching* if no vertex in H has degree more than 1. For a simple graph G, we define its *matching number* to be the maximum number of edges in any subgraph of G which is a matching.

   For each of the following graphs, compute its matching number: C_5, K_5, W_5, $K_{4,5}$.

   Solution:

   - C_5 and K_5: 2
   - W_5 : 3 (W_5 has 6 nodes, and a "perfect matching" (involving all vertices) is possible, with 2 edges as in the case of C_5, and an edge between the hub and the remaining vertex.)
   - $K_{4,5}$: 4 (More generally $K_{m,n}$ has matching number $\min\{m, n\}$.)

2. **Complement of a Graph** [8 points]

   We define the *complement of a graph* as a graph which has the same vertex set, but with exactly those edges that are absent from the original graph. Formally, if $G = (V, E)$, its complement $\overline{G} = (V, \overline{E})$, such that $\overline{E} = K_V \setminus E$ where $K_V = \{\{a, b\} | a \in V, b \in V, a \neq b\}$.

   Match each graph on the left with a description of its complement:

   (a) K_4 [a]

   (b) C_4 [d]

   (c) $K_{1,3}$ [g]

   (d) P_4 (a path with 4 nodes) [e]

   (a) A graph with no edges.

   (b) A graph with a single edge.

   (c) A path with two edges.

   (d) A matching with two edges.

   (e) A graph isomorphic to the original one.

   (f) A complete graph.

   (g) A cyclic graph.

3. **What is Wrong With this Proof?** [4 points]

Claim: If every vertex in a graph has degee at least 1, then the graph is connected.

Proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has degree at least 1, then the graph is connected.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, $P(1)$ is vacuously true.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$.

Consider an n-vertex graph G in which every vertex has degree at least 1. By the induction hypothesis, G is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to G to obtain an $(n+1)$-vertex graph H. Since x must have degree at least one, there is an edge from x to some other vertex; call it y. Since y is connected to every other node in the graph, x will be connected to every other node in the graph. QED

- ☐ A. The proof needs to consider base case $n = 2$.
- ☐ B. The proof needs to use strong induction.
- ☐ C. The proof should instead induct on the degree of each node.
- ☑ D. The proof only considers $(n+1)$ node graphs with minimum degree 1 from which deleting a vertex gives a graph with minimum degree 1.
- ☐ E. The proof only considers n node graphs with minimum degree 1 to which adding a vertex with non-zero degree gives a graph with minimum degree 1.
- ☐ F. This is a trick question. There is nothing wrong with the proof!

4. **Triangle-Free and Claw-Free Graphs.** [20 points]

Recall that an *induced subgraph* of G is obtained by removing zero or more vertices of G as well as all the edges incident on the removed vertices. (No further edges can be removed.) Formally, $G' = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{a, b\} \mid a \in V', b \in V', \{a, b\} \in E\}$.

A graph G is said to be *H-free* if no induced subgraph of G is isomorphic to H. For example, $G = (V, E)$ is K_3-free (or triangle free) if and only if there are no three distinct vertices a, b, c in V such that $\{\{a, b\}, \{b, c\}, \{c, a\}\} \subseteq E$.

Prove that the complement of a K_3-free graph is a $K_{1,3}$-free graph.[1]

[Hint: Prove the contrapositive.]

Solution: We need to prove that for any grapg G, if G is K_3-free, then $\overline{G}$ (denoting its complement) is $K_{1,3}$-free. The contrapositive states that for any graph G, if $\overline{G}$ is not $K_{1,3}$-free, then G is not K_3 free.

To prove this, suppose G is such that $\overline{G}$ is not $K_{1,3}$-free. Hence, by definition, $K_{1,3}$ is an induced subgraph of $\overline{G}$. That is, there are some 4 vertices in $\overline{G}$, say, a, b, c, d such that the edges $\{a, b\}, \{a, c\}, \{a, d\}$ are present and $\{b, c\}, \{c, d\}, \{d, b\}$ are absent in $\overline{G}$. But this means that the edges $\{b, c\}, \{c, d\}, \{d, b\}$ are present in G. Hence G is not K_3-free.

[1] The graph $K_{1,3}$ is often called the "claw" graph. So this problem can be restated as asking you to prove that the complement of a triangle-free graph is a claw-free graph.

5. **Regular Graph.** [20 points]

For any integer $n \geq 3$ and any even integer d with $2 \leq d \leq n-1$, show that there exists a d-regular graph with n nodes, by giving an explicit construction.

For full credit, describe your graph as (V, E) where $V = \mathbb{Z}_n$ and E is formally defined using modular arithmetic. (You may find it convenient to use S_a to denote $\{1, \ldots, a\} \subseteq \mathbb{Z}_n$.)

[Hint: What would you do for $d = 2$? Then consider adding additional edges for larger values of d.]

Solution: Informally, a d regular graph can can be constructed by arranging all n vertices in a circle, and joining each vertex to the first $d/2$ vertices to its right and $d/2$ vertices to its left.

Formally, let $G = (V, E)$, where $V = \mathbb{Z}_n$ and $E = \{\{a, b\} | a - b \in S_{d/2}\}$, where $S_{d/2} = \{1, \ldots, d/2\}$. Then, any vertex $a \in V$ is connected to $\{a-1, \ldots, a-d/2\} \cup \{a+1, \ldots, a+d/2\}$. Since $d \leq n-1$, the d elements in these two sets are all distinct. (To argue this more formally, note that $a - i = a + j$ only if $i + j = 0$ (in $\mathbb{Z}_n$), but since $i, j \in \{1, \ldots, d/2\}$, we have $i + j \in \{2, \ldots, d\}$, and as $d \leq n-1$, 0 does not belong to this set.)

6. **Prove using Induction.** [20 points]

Prove that for any positive integer n, for any triangle-free graph $G = (V, E)$ with $|V| = 2n$, it must be the case that $|E| \leq n^2$.

Solution:

Base case: For $n = 1$, consider any graph $G = (V, E)$, with $|V| = 2$. It can have at most one edge, and hence $|E| \leq 1 = n^2$, as claimed.

Induction step: We shall prove that for all $k \geq 1$, if the claim holds for $n = k$, it holds for $n = k+1$ as well.

Induction hypothesis: suppose for an arbitrary integer $k \geq 1$, any triangle-free graph with $2k$ nodes has at most k^2 edges.

Then, consider any triangle-free graph $G = (V, E)$ with $2(k+1) = 2k+2$ nodes.

Case 1: if G has no edge, then clearly $|E| \leq (k+1)^2$.

Case 2: if G has at least one edge, $\{u, v\}$. Then, let G' be the graph obtained from G by removing u, v and all edges incident on them. Then, by induction hypothesis, G' has at most k^2 edges. We need to count the additional edges in G (all of which involve either u or v or both). Since G is triangle-free, and the edge $\{u, v\}$ exists, there is no vertex w such that both $\{u, w\}$ and $\{v, w\}$ are edges in G (because if they did, then we will have a triangle induced by $\{u, v, w\}$. In other words, for each of the $2k$ vertices w in G', at most one of the edges $\{u, w\}$ and $\{v, w\}$ exists in G. Thus the total number of edges in G in addition to the edges in G' is at most $2k+1$ (counting at most one edge between each of the $2k$ vertices in G' and u, v, and the one edge $\{u, v\}$). Combined with the above observation that G' has at most k^2 edges, G has at most $k^2 + 2k + 1 = (k+1)^2$ edges.

7. **Prove using Strong Induction.** [20 points]

Prove that for any graph G and any two nodes a and b in G, if there is a walk from a to b, then there is a path from a to b.

[Hint: Induct on the length of the walk.]

Claim: For all $n \in \mathbb{N}$, for any graph $G = (V, E)$, and any two vertices $a, b \in V$, if G contains a walk of length n from a to b, then G has a path from a to b.

Base case: $n = 0$. In this case, any walk of length 0 from a to b (where b must be equal to a) is also a path from a to b.

Induction step: We shall prove that for any $k \in \mathbb{N}$, if the claim holds for all $n \leq k$, then it holds for $n = k + 1$.

Induction hypothesis: Suppose, for some $k \in \mathbb{N}$, for all $n \leq k$, it holds that for any graph $G = (V, E)$, and any two vertices $a, b \in V$, if G contains a walk of length n from a to b, then G has a path from a to b.

Now, suppose G is a graph with a walk W of length $k + 1$ from a to b.

Case 1: if the walk W is a path, then indeed there is a path from a to b.

Case 2: Suppose W is not a path. This means $W = a = v_0, \ldots, v_{k+1} = b$, such that for some $i < j$, $v_i = v_j$. Then consider $W' = v_0, \ldots, v_i, v_{j+1}, \ldots, v_{k+1}$. (If $j = k + 1$, $W' = v_0, \ldots, v_i$.) Note that W' is a valid path from a to b, since every two adjacent nodes in W' are adjacent in W as well (including (v_i, v_{j+1}) since $v_i = v_j$). Further, note that the length of W' is strictly smaller than that of W (since $i < j$). Thus, W' is a path of length $\ell \leq k$ from a to b, and by the induction hypothesis, there is a path from a to b.