Homework 3

Discrete Structures
CS 173 [B] : Fall 2015

Released: Thu Feb 26
Due: Thu Mar 5, 10:00 PM

Submit on Moodle.

---

1. Let $G = (V, E)$ be a bipartite graph, with two partite sets A and B (so that $V = A \cup B$, $A \cap B = \emptyset$, and $E \subseteq \{\{a, b\} \mid a \in A, b \in B\}$). Suppose we try to define a function $f : A \to B$, as follows: $f(a) = b$ if and only if $\{a, b\} \in E$.

   Match the following:

   (a) When is f well-defined?

   (b) Assuming f is well-defined, when is it onto?

   (c) Assuming f is well-defined, when is it one-to-one?

   (a) If and only if $\forall a \in A$, $\deg(a) = 1$.

   (b) If and only if $\forall a \in A$, $\deg(a) \leq 1$.

   (c) If and only if $\forall a \in A$, $\deg(a) \geq 1$.

   (d) If and only if $\forall b \in B$, $\deg(b) = 1$.

   (e) If and only if $\forall b \in B$, $\deg(b) \leq 1$.

   (f) If and only if $\forall b \in B$, $\deg(b) \geq 1$.

2. A *graph automorphism* is an isomorphism from a graph onto itself.

   For example, suppose you have a graph G with vertices labeled a, b, and c. For clarity, think of creating a copy of that graph G' with vertices labeled a', b', c' (with the corresponding edges). Then there are 3! possible bijections between the vertex sets of these two graphs, but only a subset of those mappings are isomorphisms. If G has only one edge, say, $\{a, b\}$, then there are two automorphisms f and g: $f(a) = a', f(b) = b', f(c) = c'$ and $g(a) = b', g(b) = a', g(c) = c'$. Note that f is an isomorphism from G to G', since G' indeed has the edge $\{f(a), f(b)\} = \{a', b'\}$; similarly g is an isomorphism.

   How many automorphisms does C_n, the cycle graph with n nodes, have?

   [*Hint: try solving for the automorphisms of C_3 and C_4 and find the pattern.*]

☐ A. 1

☐ B. n

☐ C. $2n$

☐ D. $n!$

☐ E. 2^n

3. Define a relation $\sim$ on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ by the rule $f \sim g$ if and only if there is a $z \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq z$. Prove that $\sim$ is an equivalence relation.

4. If functions $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is onto, then prove that g is onto. Use precise mathematical notation to prove this, starting from the definitions of onto and composition.

5. Let n be a positive integer, and $\mathbb{G}_n$ denote the set of all simple graphs on the vertex set $V = \mathbb{Z}_n$ (i.e., each vertex is labeled by an integer modulo n). Let $f : \mathbb{G}_n \to \mathbb{G}_n$ be a function defined as follows: $f((V, E)) = (V, E')$, where $E' = \{\{a, b\} \mid \{a + 1, b + 1\} \in E\}$, where the addition is modulo n.

   (a) Give examples of graphs $G_1, G_2 \in \mathbb{G}_n$ (for an n of your choice) such that $f(G_1) = G_1$ and $f(G_2) \neq G_2$.

   (b) Show that for any graph $G \in \mathbb{G}_n$, $f(G)$ is isomorphic to G.

6. Twenty-three mathematicians are eating dinner at Tang Dynasty and they have arranged to be seated at a special, unusually-large circular table. The table has a lazy Susan (central rotating circular tray) in the middle. Each person has ordered a different dish and (rather mysteriously) they all refuse to share.

   The dishes of food are brought out and placed on the lazy Susan, one dish in front of every person. However, they are entirely mismatched, so each person has another person's dish. Prove that there is a way to rotate the lazy Susan so that at least two people have the correct dish that they ordered in front of them.

   [*Hint: Imagine each mathematician holding a pigeon with a number indicating how many positions to their left their dish is.*]