Homework 3

Discrete Structures
CS 173 [B] : Fall 2015

March 6, 2015

Solutions.

1. Let $G = (V, E)$ be a bipartite graph, with two partite sets A and B (so that $V = A \cup B$, $A \cap B = \emptyset$, and $E \subseteq \{\{a, b\} \mid a \in A, b \in B\}$). Suppose we try to define a function $f : A \to B$, as follows: $f(a) = b$ if and only if $\{a, b\} \in E$.

   Match the following:

   (a) When is f well-defined? [a]

   (b) Assuming f is well-defined, when is it onto? [f]

   (c) Assuming f is well-defined, when is it one-to-one? [e]

   (a) If and only if $\forall a \in A$, $\deg(a) = 1$.

   (b) If and only if $\forall a \in A$, $\deg(a) \leq 1$.

   (c) If and only if $\forall a \in A$, $\deg(a) \geq 1$.

   (d) If and only if $\forall b \in B$, $\deg(b) = 1$.

   (e) If and only if $\forall b \in B$, $\deg(b) \leq 1$.

   (f) If and only if $\forall b \in B$, $\deg(b) \geq 1$.

2. A *graph automorphism* is an isomorphism from a graph onto itself.

   For example, suppose you have a graph G with vertices labeled a, b, and c. For clarity, think of creating a copy of that graph G' with vertices labeled a', b', c' (with the corresponding edges). Then there are 3! possible bijections between the vertex sets of these two graphs, but only a subset of those mappings are isomorphisms. If G has only one edge, say, $\{a, b\}$, then there are two automorphisms f and g: $f(a) = a', f(b) = b', f(c) = c'$ and $g(a) = b', g(b) = a', g(c) = c'$. Note that f is an isomorphism from G to G', since G' indeed has the edge $\{f(a), f(b)\} = \{a', b'\}$; similarly g is an isomorphism.

   How many automorphisms does C_n, the cycle graph with n nodes, have?

   [*Hint: try solving for the automorphisms of C_3 and C_4 and find the pattern.*]

☐ A. 1

☐ B. n

☑ C. $2n$

☐ D. $n!$

☐ E. 2^n

Solution: The automorphisms are the n "rotations," optionally followed by a "reflection." Formally, we can consider $C_n = (V, E)$, where the vertex set $V = \mathbb{Z}_n$ and $E = \{\{a, a+1\} \mid a \in \mathbb{Z}_n\}$ (note that the addition is modulo n). Then an automorphism of C_n is defined by a bijection $f : \mathbb{Z}_n \to \mathbb{Z}_n$ such that for all $a \in \mathbb{Z}_n$, $\{f(a), f(a+1)\} \in E$.

Suppose $f(0) = i$. Note that only two edges in E involving i are $\{i, i+1\}$ and $\{i, i-1\}$. So, for $\{f(0), f(1)\}$ to be in E, we need $f(1)$ to be $i-1$ or $i+1$. Once $f(0)$ and $f(1)$ are fixed, $f(2)$ is uniquely determined by the requirements that f is a bijection (hence $f(2) \neq f(0)$) and that $\{f(1), f(2)\} \in E$ (hence $f(2) = f(1) \pm 1$). Similarly, once $f(1)$ and $f(2)$ are fixed, $f(3)$ is uniquely determined. Continuing this way, by fixing $f(0)$ and $f(1)$ (to be i and $i \pm 1$), f is uniquely determined. There are n ways of choosing $f(0)$ and for each of them, 2 ways of choosing $f(1)$, leading to a total of $2n$ functions.

These functions are, for each $i \in \mathbb{Z}_n$, $f_i : \mathbb{Z}_n \to \mathbb{Z}_n$ and $g_i : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f_i(a) = i + a$ and $g_i(a) = i - a$.

3. Define a relation $\sim$ on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ by the rule $f \sim g$ if and only if there is a $z \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq z$. Prove that $\sim$ is an equivalence relation.

   Solution:

   We need to prove that $\sim$ is reflexive, symmetric and transitive.

   Reflexive: For any function $f : \mathbb{R} \to \mathbb{R}$, clearly $f \sim f$, since we can choose $z = 0$ (say), so that $f(x) = f(x)$ for all $x \geq z$.

   Symmetric: Consider any functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $f \sim g$. Then, by definition of $\sim$, there exists $z \in \mathbb{R}$ such that $\forall x \geq z$, $f(x) = g(x)$. As $=$ is symmetric, this is the same as the condition for $g \sim f$. Hence, $g \sim f$.

   Transitive: Consider any functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f \sim g$ and $g \sim h$. Then there are real numbers z_1 and z_2 such that $\forall x \geq z_1$, $f(x) = g(x)$, and $\forall x \geq z_2$, $g(x) = h(x)$. Let $z = \max z_1, z_2$. Then, $\forall x \geq z$, we have both $f(x) = g(x)$ and $g(x) = h(x)$, and hence $f(x) = h(x)$. Hence, $f \sim h$.

4. If functions $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is onto, then prove that g is onto. Use precise mathematical notation to prove this, starting from the definitions of onto and composition.

   Solution:

   Suppose functions $f : A \to B$ and $g : B \to C$ are such that $g \circ f : A \to C$ is onto. Then, we have that $\forall y \in C$, $\exists x \in A$, such that $g(f(x)) = y$. But for any $x \in A$, $f(x) \in B$. Hence, we have $\forall y \in C$, $\exists x \in A, z \in B$, such that $z = f(x)$ and $g(z) = y$. In particular, $\forall y \in C$, $\exists z \in B$, such that $g(z) = y$. But this is the definition of g being onto. Hence if $g \circ f$ is onto, g is onto.

5. Let n be a positive integer, and $\mathbb{G}_n$ denote the set of all simple graphs on the vertex set $V = \mathbb{Z}_n$ (i.e., each vertex is labeled by an integer modulo n). Let $f : \mathbb{G}_n \to \mathbb{G}_n$ be a function defined as follows: $f((V, E)) = (V, E')$, where $E' = \{\{a, b\} \mid \{a+1, b+1\} \in E\}$, where the addition is modulo n.

   (a) Give examples of graphs G_1 and G_2 such that $f(G_1) = G_1$ and $f(G_2) \neq G_2$.

   Solution: Here are some examples of G_1 such that $f(G_1) = G_1$.

   - G_1 be the graph with no edges: $E = \emptyset$.
   - G_1 be the graph with $E = \{\{a, a+i\} \mid a \in \mathbb{Z}_n\}$ for some i. (In particular, with $i = 1$, G_1 will be the cycle graph).
   - G_1 can be a graph whose edge-set is the union of the edge-sets of any of the above graphs. (In particular, G_1 can be the complete graph.)

Any other graph would in fact be an example for G_2 such that $f(G_2) \neq G_2$. In particular, consider G_2 to have edge-set $E = \{\{0,1\}\}$ (assuming $n > 2$).

(b) Show that for any graph $G \in \mathbb{G}_n$, $f(G)$ is isomorphic to G.

Solution: Consider the function $g : \mathbb{Z}_n \to \mathbb{Z}_n$ defined as $g(x) = x+1$ (where the addition is modulo n). Firstly, g is a bijection: for any $y \in \mathbb{Z}_n$, $g(x) = y$ if and only if $x = y-1$, and hence y has exactly one pre-image. Secondly, $\{a, b\}$ is an edge in $f(G)$ if and only if $\{\{a+1, b+1\}\} = \{g(a), g(b)\}$ in an edge in G. Hence g is an isomorphism from $f(G)$ to G. (Equivalently, g^{-1} defined as $g^{-1}(x) = x - 1$ is an isomorphism from G to $f(G)$.) Hence G and $f(G)$ are isomorphic to each other.

6. Twenty-three mathematicians are eating dinner at Tang Dynasty and they have arranged to be seated at a special, unusually-large circular table. The table has a lazy Susan (central rotating circular tray) in the middle. Each person has ordered a different dish and (rather mysteriously) they all refuse to share.

The dishes of food are brought out and placed on the lazy Susan, one dish in front of every person. However, they are entirely mismatched, so each person has another person's dish. Prove that there is a way to rotate the lazy Susan so that at least two people have the correct dish that they ordered in front of them.

[*Hint: Imagine each mathematician holding a pigeon with a number indicating how many positions to their left their dish is.*]

Solution: As given in the hint, consider a function that maps each mathematician to a number indicating how many positions to their left their dish is. Since no mathematician has their dish in front of them, we know that this number is never 0. Further, since there are only n mathematicians ($n = 23$), there are $n - 1$ positions to each person's left (not counting their own position). Thus each of the n mathematicians is assigned a value in the set $\{1, \dots, n-1\}$. This means, at least two of them, say Alice and Bob, are assigned the same number, say x. Then, by rotating the table x positions to the right (as seen by each mathematician), Alice and Bob both will have their dish in front of them.