Homework 1

Discrete Structures
CS 173 [B] : Fall 2015

Released: Tue Jan 26
Due: Thu Feb 5, 10:00 PM

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1. **Simplifying formulas.** [20 points]

   Every formula in two variables corresponds to a binary operator. Identify the operator in the following cases, and write down an equivalent expression.

   (Thus your answer should be one of the 16 possibilities: T, F, p, q, $\neg p$, $\neg q$, $p \oplus q$, $p \leftrightarrow q$, $p \wedge q$, $p \vee q$, $p \uparrow q$, $p \downarrow q$, $p \rightarrow q$, $q \rightarrow p$, $p \not\rightarrow q$ and $q \not\rightarrow p$.)

   You may prepare a truth table for each formula to help with the task. You could also employ the distributive property, De Morgan's law and other equivalences from the lecture.

   (a) $(p \rightarrow q) \wedge \neg q$
   (b) $p \vee \neg(q \rightarrow p)$
   (c) $(p \wedge q) \rightarrow q$
   (d) $(p \wedge q) \leftrightarrow q$
   (e) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q) \vee (p \wedge \neg p))$

2. **Predicate logic. In plain English.** [16 points]

   Suppose we define the following predicates, over the domain of living things on earth.

   $H(x)$ tells if x is a human (i.e., $H(x) = T$ if and only if x is human).

   $R(x)$ tells if x is a "reptilian."

   $S(x)$ tells if x is from outer space.

   $C(x)$ tells if x is a cow.

   $A(x, y)$ tells if x abducts y.

   $I(x, y)$ tells if x is more intelligent than y.

   Translate each of the following into English.

   (a) $\exists x\ H(x) \wedge R(x)$
   (b) $\forall y\ R(y) \rightarrow S(y)$
   (c) $\exists x\ S(x) \wedge (\exists y\ C(y) \wedge A(x, y))$
   (d) $\forall x \exists y\ C(x) \rightarrow (H(y) \wedge I(x, y))$

3. How many ternary logical operators are there? [4 points]

4. **Functional Completeness.** [20 points]

   A set of operators is *functionally complete* if all n-ary logical operations, for any $n > 0$, can be expressed as formulas that use only operators from this set. In other words, all possible truth tables *over any number of inputs* can be produced by formulas that use only these operators.

   (a) Show that the set $\{\neg, \wedge, \vee\}$ is functionally complete.
   [*Hint: First consider an n-ary operation which has a single row in its truth table evaluating to T. Can you design an equivalent formula with just $\neg$s and $\wedge$s? Next, if an operation's truth table has k rows that evaluate to T, can you design a formula with k terms of the above kind, combined using $\vee$s?*]

   (b) Is the set $\{\neg, \vee\}$ functionally complete? Explain why or why not.
   [*Hint: Can you express $p \wedge q$ using only $\neg$ and $\vee$?*]

5. Is the following argument valid? Explain. [10 points]

   - If my house is less than a mile away from my office, I walk to work.
   - I walk to work.
   - Therefore, my house is less than a mile away from my office.

   [*Hint: Denote the proposition "my house is less than a mile away from my office" by p, and the proposition "I walk to work" by q. Then write down the proposition that corresponds to the AND of first two items above. Does it "imply" the last one?*]

6. **A Tautology.** [15 points]

   Prove that $\exists x \forall y\ P(x) \rightarrow P(y)$ is true no matter what the predicate P is (assuming that the domain is non-empty).

   [Hint: consider two cases, depending on whether $\forall y\ P(y)$ is true or false.]

7. **Intervals.** [15 points]

   A pair of real numbers (x, y) is said to be an *interval* if $x \leq y$. An interval (x, y) is said to *contain* an interval (p, q) if $x \leq p$ and $q \leq y$. Using this definition, prove or disprove the following:

   (a) For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (c, d) contains (e, f), then (a, b) contains (e, f).

   (b) For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (a, b) contains (e, f), then either (c, d) contains (e, f) or (e, f) contains (c, d) (or both).