Homework 1

Discrete Structures
CS 173 [B] : Fall 2015

Released: Tue Jan 26
Due: Thu Feb 5, 10:00 PM

Solutions for Manually Graded Problems.

1. **Functional Completeness.** [20 points]

A set of operators is *functionally complete* if all n-ary logical operations, for any $n > 0$, can be expressed as formulas that use only operators from this set. In other words, all possible truth tables *over any number of inputs* can be produced by formulas that use only these operators.

(a) Show that the set $\{\neg, \wedge, \vee\}$ is functionally complete.
[*Hint: First consider an n-ary operation which has a single row in its truth table evaluating to T. Can you design an equivalent formula with just $\neg$s and $\wedge$s? Next, if an operation's truth table has k rows that evaluate to T, can you design a formula with k terms of the above kind, combined using $\vee$s?*]

Solution:
Consider an arbitrary n-ary logical operation op, for an arbitrary integer $n > 0$. We shall construct a formula for $\mathsf{op}(X_1, \ldots, X_n)$.
Let N denote the number of rows in the truth table of op which evaluate to T. Let the i^{th} such row be indexed by a vector $(\alpha_{i,1}, \ldots, \alpha_{i,n}) \in \{T, F\}^n$, such that $\mathsf{op}(\alpha_{i,1}, \ldots, \alpha_{i,n}) = T$. Then, for any vector $(x_1, \ldots, x_n) \in \{T, F\}^n$, we have that $\mathsf{op}(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n) \in \{(\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, (\alpha_{N,1}, \ldots, \alpha_{N,n})\}$.
Now, we construct a formula for op. For each $i \in \{1, \ldots, N\}$, for each $j \in \{1, \ldots, n\}$, define:

$$F_{i,j}(X_1, \ldots, X_n) \equiv \begin{cases} X_j & \text{if } \alpha_{i,j} = T \\ \neg X_j & \text{if } \alpha_{i,j} = F \end{cases}$$

Note that $F_{i,j}(x_1, \ldots, x_n) = T$ iff $x_j = \alpha_{i,j}$. For each $i \in \{1, \ldots, N\}$, let

$$G_i(X_1, \ldots, X_n) \equiv F_{i,1}(X_1, \ldots, X_n) \wedge \cdots \wedge F_{i,n}(X_1, \ldots, X_n).$$

Note that $G_i(x_1, \ldots, x_n) = T$ if and only if $(x_1, \ldots, x_n) = (\alpha_{i,1}, \ldots, \alpha_{i,n})$.
Finally, let

$$F(X_1, \ldots, X_n) \equiv G_1(X_1, \ldots, X_n) \vee \cdots \vee G_N(X_1, \ldots, X_n).$$

We note that $F(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n) \in \{(\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, (\alpha_{N,1}, \ldots, \alpha_{N,n})\}$. Also, as noted above $\mathsf{op}(x_1, \ldots, x_n) = T$ iff $(x_1, \ldots, x_n)$ belongs to the same set. Thus $\mathsf{op}(X_1, \ldots, X_n) \equiv F(X_1, \ldots, X_n)$
As F uses only the operators $\wedge$, $\vee$ and $\neg$, and since op could be any n-ary operator for any $n > 0$, the set $\{\wedge, \vee, \neg\}$ is functionally complete.

(b) Is the set $\{\neg, \vee\}$ functionally complete? Explain why or why not.
[*Hint: Can you express* $p \wedge q$ *using only* $\neg$ *and* $\vee$*?*]

Solution:

The set $\{\neg, \vee\}$ is functionally complete.

Since $\{\neg, \wedge, \vee\}$ is functionally complete, any n-ary operator has a formula F involving only these operators. Further, since, $p \wedge q \equiv \neg(\neg p \vee \neg q)$, F is equivalent to a formula in which we recursively replace each instance of $\wedge$ using the above equivalence.

Alternately, in the previous derivation, replace the definition of G_i with

$$G_i(X_1, \ldots, X_n) \equiv \neg\big(\neg F_{i,1}(X_1, \ldots, X_n) \vee \cdots \vee \neg F_{i,n}(X_1, \ldots, X_n)\big).$$

2. Is the following argument valid? Explain. [10 points]

- If my house is less than a mile away from my office, I walk to work.
- I walk to work.
- Therefore, my house is less than a mile away from my office.

[*Hint: Denote the proposition "my house is less than a mile away from my office" by p, and the proposition "I walk to work" by q. Then write down the proposition that corresponds to the AND of first two items above. Does it "imply" the last one?*]

Solution: The argument is not valid. If we let $p =$ my house is less than a mile away from my office, and $q =$ I walk to work, then we have the following argument:

- $p \to q$
- q
- $\therefore p$

If $p = F$ while $q = T$, then $p \to q = T$ and $q = T$, but $p = F$. In other words, there is an assignment where the premises are true and the conclusion is false.

3. **A Tautology.** [15 points]

Prove that $\exists x \forall y\; P(x) \to P(y)$ is true no matter what the predicate P is (assuming that the domain is non-empty).

[Hint: consider two cases, depending on whether $\forall y\; P(y)$ is true or false.]

Solution 1:

There are two possible cases
Case 1: $\forall y P(y)$ is true.

- Since the domain is non-empty, there exists at least one element in the domain, let's say w.
- Note that $P(w) \to P(y)$ for every y since, $P(w)$ is true and $P(y)$ is true for all y.
- Hence, $(\forall y P(w) \to P(y))$ is true.
- From this we can conclude that $\exists x \forall y P(x) \to P(y)$ is true.

Case 2:

- $\forall y P(y)$ is false which means $\neg(\forall y P(y))$ is true.

- $\neg(\forall y P(y)) \equiv \exists y \neg P(y)$ is true. Let a be the element such that $\neg P(a)$ is true. Then $P(a)$ is false.
- Since $P(a)$ is false, $P(a) \rightarrow P(y)$ is true for any y. That is, $\forall y,\ P(a) \rightarrow P(y)$ is true.
- Since, $\forall y,\ P(a) \rightarrow P(y)$ is true, $\exists x \forall y,\ P(x) \rightarrow P(y)$ is true (by considering x to be a).

Solution 2:

$$\begin{aligned}
\exists x(\forall y(P(x) \rightarrow P(y))) &\equiv \exists x(\forall y(\neg P(x) \lor P(y))) && \text{using the rule } (p \rightarrow q) \equiv (\neg p \lor q) \\
&\equiv \exists x(\neg P(x) \lor (\forall y P(y))) && \text{using the rule } \forall y\ R \lor Q(y) \equiv R \lor \forall y\ Q(y) \\
&\equiv (\exists x \neg P(x)) \lor (\forall y P(y)) && \text{using the rule } \exists x\ (P(x) \lor R) \equiv (\exists x\ P(x)) \lor R \\
&\equiv \neg(\forall x P(x)) \lor (\forall y P(y)) && \text{using the rule } \neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x) \\
&\equiv \neg(\forall y P(y)) \lor (\forall y P(y)) && \text{using the rule } \forall x\ P(x) \equiv \forall\ y P(y) \\
&\equiv T && \text{using the rule } \neg p \lor p \equiv T
\end{aligned}$$

Thus we can conclude that $\exists x \forall y P(x) \rightarrow P(y) \equiv T$. Hence proved.

4. **Intervals.** [15 points]

A pair of real numbers (x, y) is said to be an *interval* if $x \leq y$. An interval (x, y) is said to *contain* an interval (p, q) if $x \leq p$ and $q \leq y$. Using this definition, prove or disprove the following:

(a) For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (c, d) contains (e, f), then (a, b) contains (e, f).

(b) For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (a, b) contains (e, f), then either (c, d) contains (e, f) or (e, f) contains (c, d) (or both).

Solution:

(a) True.

Proof:

- (a, b) contains (c, d) implies that $a \leq c$ and $d \leq b$
- (c, d) contains (e, f) implies that $c \leq e$ and $f \leq d$.
- Since $a \leq c$ and $c \leq e$, we have $a \leq e$. Since $f \leq d$ and $d \leq b$, we have $f \leq b$.
- Since $a \leq e$ and $f \leq b$, we can conclude that (a, b) contains (e, f).

(b) False.

Counterexample: Consider $a = 0, b = 5, c = 1, d = 2, e = 3, f = 4$.

For this example, (a, b) contains (c, d) and (e, f) but neither does (c, d) contain (e, f) nor does (e, f) contain (c, d). Hence, statement "if (a, b) contains (c, d) and (e, f) then either (c, d) contains (e, f) or (e, f) contains (c, d)" is false.