CS 173 (B), Spring 2015, Examlet 6, Part A

NAME:	NETID:

Discussion Section: BDA:1PM BDB:2PM BDC:3PM BDD:4PM BDE:5PM

In the following problems, you can leave your answer as an arithmetic expression (involving addition, multiplication, exponentiation and factorial), unless otherwise specified.

1. **Rent-a-T.Ag** [10 points]

   A start-up called T.Ag rents out short tags that last for a week. The tags are 4 characters long, and are made up of the 26 lower-case letters and 10 digits (i.e., four characters from the set $\{\mathtt{a}, \ldots, \mathtt{z}, \mathtt{0}, \ldots, \mathtt{9}\}$).

   T.Ag charges the users a few cents for owning a tag for a week (which will allow them to own a webpage with a URL of the form `http://t.ag/abcd`, where instead of `abcd` the tag appears). The rates for the different tags are as follows.

   - 1¢ for a tag which uses only digits.
   - 2¢ for a tag which uses only letters.
   - 3¢ for a tag which uses both letters and digits.

   What is the maximum weekly revenue that T.Ag can generate from tag-rentals, assuming insatiable demand?

   (No two customers can have the same tag in the same week.)

   Solution: Maximum weekly revenue from tag-rentals is $1 \cdot 10^4 + 2 \cdot 26^4 + 3 \cdot (36^4 - 26^4 - 10^4)$ ¢$= 3 \cdot 36^4 - 26^4 - 2 \cdot 10^4$ ¢.

2. In how many ways can 12 chairs — 4 white chairs, 4 black chairs and 4 red chairs — be arranged in a row? (Chairs of the same color are indistinguishable from each other.) [5 points]

Solution:

Positions for the 4 white chairs can be chosen in $\binom{12}{4}$ ways, and then from among the remaining 8 positions, the positions for the 4 black chairs can be chosen in $\binom{8}{4}$ ways. So in all, the total number of ways is

$$\binom{12}{4} \times \binom{8}{4} = \frac{12!}{8!4!} \cdot \frac{8!}{4!4!} = \frac{12!}{4!4!4!}.$$

3. **Smooth Sequences** [10 points]

A sequence of integers is said to be *smooth* if any two consecutive integers in the sequence differ by exactly 1. For instance, $5, 4, 5, 6, 5, 4$ is a smooth sequence of length 6.

How many smooth sequences of length 16 are there that start with 10 and end with 15?

Solution:

Consider the differences between the consecutive integers in a smooth sequence of length n. This is a sequence of ± 1 of length $n-1$. Once the first number in the smooth sequence is fixed, then the difference-sequence fully determines the smooth sequence. Further, the number of $+1$'s and -1's in the difference sequence uniquely determines the last number in the smooth sequence, and vice versa. Hence, to count the number of smooth sequences of length n with a given first and last number, it is enough to count the number of difference-sequences of length $n-1$ with the appropriate number of $+1$'s and -1's.

Formally, given a smooth sequence $x_1, \ldots, x_n$, let $d_1, \ldots, d_{n-1}$ be defined as $d_i = x_{i+1} - x_i$, where each $d_i \in \{1, -1\}$. Further, $\sum_{i=1}^{n-1} d_i = x_n - x_1$.

Let a denote the number of $+1$'s in the sequence, and b denote the number of -1's: i.e., $a = |\{i : d_i = 1\}|$ and $b = |\{i : d_i = -1\}|$.

Then $a + b = n - 1$. Also, $\sum_{i=1}^{n-1} d_i = a - b$. That is, $a - b = x_n - x_1$. Thus $a = \frac{1}{2}(n - 1 + x_n - x_1)$.

The number of difference sequences of length $n-1$ with a $+1$'s and b -1's is $\binom{n-1}{a}$.

In the given problem $n = 16$, and $x_n - x_1 = 5$. Hence $a = \frac{1}{2}(15 + 5) = 10$. Hence the answer is $\binom{15}{10} = \frac{15!}{10!5!}$.

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1. **Rent-a-T.Ag** [10 points]

   A start-up called T.Ag rents out short tags that last for a week. The tags are 4 characters long, and are made up of the 26 lower-case letters and 10 digits (i.e., four characters from the set $\{\texttt{a}, \ldots, \texttt{z}, \texttt{0}, \ldots, \texttt{9}\}$).

   T.Ag charges the users a few cents for owning a tag for a week (which will allow them to own a webpage with a URL of the form `http://t.ag/abcd`, where instead of `abcd` the tag appears). The rates for the different tags are as follows.

   - 4¢ for a tag which uses only letters.
   - 2¢ for a tag which uses only digits.
   - 1¢ for a tag which uses both letters and digits.

   What is the maximum weekly revenue that T.Ag can generate from tag-rentals, assuming insatiable demand?

   (No two customers can have the same tag in the same week.)

   Maximum weekly revenue from tag-rentals is $4 \cdot 26^4 + 2 \cdot 10^4 + 1 \cdot (36^4 - 26^4 - 10^4)$ ¢$= 3 \cdot 26^4 + 10^4 + 36^4$ ¢.

2. In how many ways can 12 chairs — 5 white chairs, 4 black chairs and 3 red chairs — be arranged in a row? (Chairs of the same color are indistinguishable from each other.) [5 points]

Solution:

Positions for the 5 white chairs can be chosen in $\binom{12}{5}$ ways, and then from among the remaining 7 positions, the positions for the 4 black chairs can be chosen in $\binom{7}{4}$ ways. So in all, the total number of ways is

$$\binom{12}{5} \times \binom{7}{4} = \frac{12!}{5!7!} \cdot \frac{7!}{4!3!} = \frac{12!}{5!4!3!}.$$

3. **Smooth Sequences** [10 points]

A sequence of integers is said to be *smooth* if any two consecutive integers in the sequence differ by exactly 1. For instance, $5, 4, 5, 6, 5, 4$ is a smooth sequence of length 6.

How many smooth sequences of length 15 are there that start with 12 and end with 18?

Solution:

Consider the differences between the consecutive integers in a smooth sequence of length n. This is a sequence of ± 1 of length $n-1$. Once the first number in the smooth sequence is fixed, then the difference-sequence fully determines the smooth sequence. Further, the number of $+1$'s and -1's in the difference sequence uniquely determines the last number in the smooth sequence, and vice versa. Hence, to count the number of smooth sequences of length n with a given first and last number, it is enough to count the number of difference-sequences of length $n-1$ with the appropriate number of $+1$'s and -1's.

Formally, given a smooth sequence $x_1, \ldots, x_n$, let $d_1, \ldots, d_{n-1}$ be defined as $d_i = x_{i+1} - x_i$, where each $d_i \in \{1, -1\}$. Further, $\sum_{i=1}^{n-1} d_i = x_n - x_1$.

Let a denote the number of $+1$'s in the sequence, and b denote the number of -1's: i.e., $a = |\{i : d_i = 1\}|$ and $b = |\{i : d_i = -1\}|$.

Then $a + b = n - 1$. Also, $\sum_{i=1}^{n-1} d_i = a - b$. That is, $a - b = x_n - x_1$. Thus $a = \frac{1}{2}(n - 1 + x_n - x_1)$.

The number of difference sequences of length $n-1$ with a $+1$'s and b -1's is $\binom{n-1}{a}$.

In the given problem $n = 15$, and $x_n - x_1 = 6$. Hence $a = \frac{1}{2}(14 + 6) = 10$. Hence the answer is $\binom{14}{10} = \frac{14!}{10!4!}$.