CS 173 (B), Spring 2015, Examlet 4, Part A

NAME:	**NETID:**

Discussion Section: BDA:1PM BDB:2PM BDC:3PM BDD:4PM BDE:5PM

1. Below is a proof by strong induction that, for any $n \in \mathbb{Z}^+$, any connected graph with n nodes has at least $n-1$ edges. The induction variable is n.

   Fill in the blanks below to complete the proof. [25 points]

   (a) The base case claim, which is clearly true, is that

   _____every connected graph with 1 node has at least 0 edges_____.

   (b) The induction step. We claim that:

   (*range for k:*) $\forall k \geq$ __2__, if

   (*induction hypothesis:*) $\forall n \in \mathbb{Z}^+$ such that $n \leq k-1$, it holds that any connected graph G with n nodes has ___at least $n-1$___ edges,

   (*to prove:*) then any connected graph G with k nodes has at least $k-1$ edges.

   (c) To prove the above claim, consider an arbitrary connected graph G with ___k___ nodes. Let m denote the number of edges in G. We need to prove that m ___$\geq k-1$___.

   Let u be an arbitrary node in G. Let d be the degree of u. From G, if we remove u and the d edges connected to it, we obtain a subgraph H of G.

   Let t denote the number of connected components in H. Then (give an upper bound)

   $$t \underline{\quad \leq d \quad} \tag{1}$$

   (Justification omitted.)

For each $i = 1, \ldots, t$, let n_i denote the number of vertices in the i^{th} connected component of H, and m_i be the number of edges in it. Clearly,

$$\sum_{i=1}^{t} n_i = \underline{\quad k-1 \quad} \tag{2}$$

Also, for each $i = 1, \ldots, t$, since $n_i \leq k-1$, by the induction hypothesis (relate n_i and m_i):

$$\underline{\quad m_i \geq n_i - 1 \quad} \tag{3}$$

Finally, the number of edges in G, $m = \underline{\quad d + \sum_{i=1}^{t} m_i \quad}$ (relate it to m_i). Hence, by equations (1), (2) and (3), (complete the proof)

$$\underline{\quad m \geq d + \sum_{i=1}^{t} (n_i - 1) \quad}$$

$$\underline{\quad = d + (k-1) - t \geq k-1 \quad}.$$

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   (a) The base case claim, which is clearly true, is that

   ______every connected graph with 1 node has at least 0 edges______.

   (b) The induction step. We claim that:

   (*range for k:*) $\forall k \geq$ __1__, if

   (*induction hypothesis:*) $\forall n \in \mathbb{Z}^+$ such that $n \leq k$, it holds that any connected graph G with n nodes has ____at least $n-1$____ edges,

   (*to prove:*) then any connected graph G with $k+1$ nodes has at least k edges.

   (c) To prove the above claim, consider an arbitrary connected graph G with ____$k+1$____ nodes. Let m denote the number of edges in G. We need to prove that m ____$\geq k$____.

   Let u be an arbitrary node in G. Let d be the degree of u. From G, if we remove u and the d edges connected to it, we obtain a subgraph H of G.

   Let t denote the number of connected components in H. Then (give an upper bound)

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$$\underline{\qquad m \geq d + \sum_{i=1}^{t} (n_i - 1) \qquad}$$

$$\underline{\qquad = d + k - t \geq k \qquad}.$$