CS 173 (B), Spring 2015, Examlet 3, Part A

NAME:	NETID:

Discussion Section: BDA:1PM BDB:2PM BDC:3PM BDD:4PM BDE:5PM

1. Given a function f, and a relation $\sim$ over the co-domain of f, let $\approx_f$ be a relation over the domain of f defined as follows: $a \approx_f b$ if $f(a) \sim f(b)$.

   (a) Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = \lfloor x/5 \rfloor$. Let $\sim$ be $=$ (equality). Then list all $a \in \mathbb{Z}$ such that $a \approx_f 3$. [5 points]

   Solution: $f(3) = 0$. We have $a \approx_f 3$, or equivalently $f(a) = f(3)$, and for all $a \in \{0, 1, 2, 3, 4\}$.

   - ♠ Full points if the list is correct (even without any other work).
   - ♠ 4 points if 0 is omitted or 5 is added to the list (or both).
   - ♠ 2 points for just getting $f(3) = 0$.
   - ♠ 1 point for a list that includes 3.

   (b) Prove that, for any f, if $\sim$ is an equivalence relation, then $\approx_f$ is an equivalence relation. (You should explicitly prove that it satisfies all the properties required of an equivalence relation.) [10 points]

   Solution: We shall prove that $\approx_f$ is reflexive, symmetric and transitive:

   Reflexive: Consider an arbitrary element $a \in \mathbb{Z}$. We have $f(a) \sim f(a)$, since $\sim$ is reflexive. Hence, by definition of $\approx_f$, we have $a \approx_f a$. Since this holds for any $a \in \mathbb{Z}$, $\approx_f$ is reflexive.

   Symmetric: Consider arbitrary $a, b \in \mathbb{Z}$. Suppose $a \approx_f b$. Then, by definition of $\approx_f$, $f(a) \sim f(b)$. Since $\sim$ is symmetric, this implies that $f(b) \sim f(a)$. Hence, by definition of $\approx_f$, $b \approx_f a$. Thus for any $a, b \in \mathbb{Z}$ such that $a \approx_f b$, we have $b \approx_f a$. Hence $\approx_f$ is symmetric.

   Transitive: Consider arbitrary $a, b, c \in \mathbb{Z}$. Suppose $a \approx_f b$ and $b \approx_f c$. Then, by definition of $\approx_f$, we have $f(a) \sim f(b)$ and $f(b) \sim f(c)$. Since $\sim$ is transitive, we have $f(a) \sim f(c)$. Then, applying the definition of $\approx_f$ again, we have $a \approx_f c$. Thus for any $a, b, c \in \mathbb{Z}$, $a \approx_f b$ and $b \approx_f c$, we have $a \approx_f c$. Hence $\approx_f$ is transitive.

   - ♠ 4 points for just naming the 3 properties to prove.
   - ♠ 2 points each for the proof of each property.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f((x, y)) = (y, y - x)$. Then define f^{-1}, or show that there is no unique inverse for f. [6 points]

Solution: To invert f, given $f((x, y)) = (a, b)$, we need to solve for (x, y). That is, we need to solve for (x, y) from $(y, y - x) = (a, b)$. That is, $y = a$ and $y - x = b$. Hence $x = y - b = a - b$. Thus, given $(y, y - x) = (a, b)$, the unique solution is $(x, y) = (a - b, b)$. Hence,

$$f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is defined as } f((a, b)) = (a - b, b).$$

♠ 3 points if solving for (x, y) is mentioned, even if they conclude that no unique inverse exists.

3. Definitions (use mathematical notation involving quantifiers $\forall$ and $\exists$ only): [4 points]

$f : A \to B$ is said to be onto if:

Solution:

$\forall y \in B, \exists x \in A$ such that $f(x) = y$.

♠ 2 points for this part.
♠ 1 point if quantifiers reversed.

$f : A \to B$ is said to be one-to-one if:

Solution:

$\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \to x_1 = x_2$.

♠ 2 points for this part.
♠ 2 points if using $\exists!$ and $\text{Im}(f)$: $\forall y \in \text{Im}(f), \exists! x \in A,\ f(x) = f(y)$.
♠ 2 points for contrapositive (or an equivalent statement): $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \to f(x_1) \neq f(x_2)$.
Or, "for all distinct $x_1, x_2 \in A$, $f(x_1) \neq f(x_2)$."
♠ 1 point if "distinct" is missing from above.

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1. Given a function f, and a relation $\sim$ over the co-domain of f, let $\approx_f$ be a relation over the domain of f defined as follows: $a \approx_f b$ if $f(a) \sim f(b)$.

   (a) Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = \lfloor x/4 \rfloor$. Let $\sim$ be $=$ (equality). Then list all $a \in \mathbb{Z}$ such that $a \approx_f 6$. [5 points]

   Solution: $f(6) = 1$. We have $a \approx_f 6$, or equivalently $f(a) = f(6)$, and for all $a \in \{4, 5, 6, 7\}$.

   - ♠ Full points if the list is correct (even without any other work).
   - ♠ 4 points if 4 is omitted or 8 is added to the list (or both).
   - ♠ 2 points for just getting $f(6) = 0$.
   - ♠ 1 point for a list that includes 6.

   (b) Prove that, for any f, if $\sim$ is an equivalence relation, then $\approx_f$ is an equivalence relation. (You should explicitly prove that it satisfies all the properties required of an equivalence relation.) [10 points]

   Solution: We shall prove that $\approx_f$ is reflexive, symmetric and transitive:
   <u>Reflexive:</u> Consider an arbitrary element $a \in \mathbb{Z}$. We have $f(a) \sim f(a)$, since $\sim$ is reflexive. Hence, by definition of $\approx_f$, we have $a \approx_f a$. Since this holds for any $a \in \mathbb{Z}$, $\approx_f$ is reflexive.
   <u>Symmetric:</u> Consider arbitrary $a, b \in \mathbb{Z}$. Suppose $a \approx_f b$. Then, by definition of $\approx_f$, $f(a) \sim f(b)$. Since $\sim$ is symmetric, this implies that $f(b) \sim f(a)$. Hence, by definition of $\approx_f$, $b \approx_f a$. Thus for any $a, b \in \mathbb{Z}$ such that $a \approx_f b$, we have $b \approx_f a$. Hence $\approx_f$ is symmetric.
   <u>Transitive:</u> Consider arbitrary $a, b, c \in \mathbb{Z}$. Suppose $a \approx_f b$ and $b \approx_f c$. Then, by definition of $\approx_f$, we have $f(a) \sim f(b)$ and $f(b) \sim f(c)$. Since $\sim$ is transitive, we have $f(a) \sim f(c)$. Then, applying the definition of $\approx_f$ again, we have $a \approx_f c$. Thus for any $a, b, c \in \mathbb{Z}$, $a \approx_f b$ and $b \approx_f c$, we have $a \approx_f c$. Hence $\approx_f$ is transitive.

   - ♠ 4 points for just naming the 3 properties to prove.
   - ♠ 2 points each for the proof of each property.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f((x, y)) = (x - y, x)$. Then define f^{-1}, or show that there is no unique inverse for f. [6 points]

Solution: To invert f, given $f((x, y)) = (a, b)$, we need to solve for (x, y). That is, we need to solve for (x, y) from $(x - y, x) = (a, b)$. That is, $x - y = a$ and $x = b$. Hence $y = x - a = b - a$. Thus, given $(x - y, x) = (a, b)$, the unique solution is $(x, y) = (b, b - a)$. Hence,

$$f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is defined as } f((a, b)) = (b, b - a).$$

♠ 3 points if solving for (x, y) is mentioned, even if they conclude that no unique inverse exists.

3. Definitions (use mathematical notation involving quantifiers $\forall$ and $\exists$ only): [4 points]

$f : A \to B$ is said to be one-to-one if:

Solution:

$\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \to x_1 = x_2$.

♠ 2 points for this part.
♠ 2 points if using $\exists!$ and $\text{Im}(f)$: $\forall y \in \text{Im}(f), \exists! x \in A,\ f(x) = f(y)$.
♠ 2 points for contrapositive (or an equivalent statement): $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \to f(x_1) \neq f(x_2)$.
Or, "for all distinct $x_1, x_2 \in A$, $f(x_1) \neq f(x_2)$."
♠ 1 point if "distinct" is missing from above.

$f : A \to B$ is said to be onto if:

Solution:

$\forall y \in B, \exists x \in A$ such that $f(x) = y$.

♠ 2 points for this part.
♠ 1 point if quantifiers reversed.