CS 173, Fall 2012
Midterm 2 Solutions

Problem 1: Checkbox (14 points)

Check the box that best characterizes each item.

$\exists x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ x \geq y$	**true** ☐ **false** ☑	
I found 143 marbles in my saucepan last Saturday. 143 is _____ the number of marbles that fits in my saucepan	**exactly** ☐ **a lower bound on** ☑ **an upper bound on** ☐	
Number of nodes at level k in a full complete binary tree.	2^k ☑ $2^k - 1$ ☐ $2^{k+1} - 1$ ☐ 2^{k-1} ☐	
$f : \mathbb{Z} \to \mathbb{R}$ $f(x) = 2x$ The codomain of f is	$\mathbb{Z}$ ☐ $\mathbb{R}$ ☑ {**even integers**} ☐ $2x$ ☐	
The shortest possible cycles have	**0 nodes** ☐ **1 node** ☐ **2 nodes** ☐ **3 nodes** ☑	The definition of cycle explicitly requires at least 3 nodes.

The diameter of the wheel graph W_5	**1** ☐	**2** $\surd$	**3** ☐
	4 ☐	**5** ☐	**6** ☐

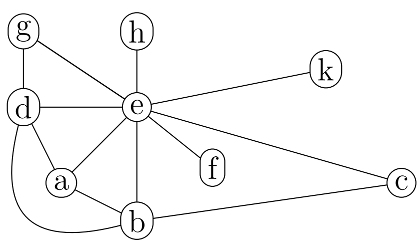
The number of edges in K_n (complete graph on n nodes)	n ☐	$\frac{n(n-1)}{2}$ $\surd$
	$\frac{n(n+1)}{2}$ ☐	$\frac{n}{2}$ ☐

Problem 2: Short answer (15 points)

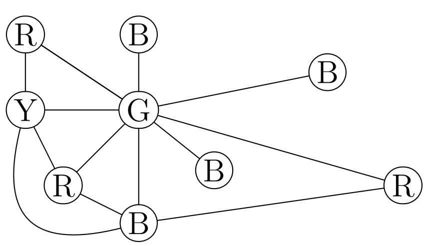
(a) (4 points) How many different 11-letter strings can I make by permuting the letters in the 11-letter word "confessions"?

Solution: The repeated letters are: 2 copies of o, two copies of n, three copies of s. So the number of strings is $\frac{11!}{2 \cdot 2 \cdot 3!}$.

(b) (6 points) What is the chromatic number of graph G (below)? Justify your answer.



Solution: The chromatic number is 4. The picture below shows how to color the graph with 4 colors (an upper bound). And we know that at least 4 colors are required (a lower bound) because the graph contains a copy of K_4, i.e. using the nodes a, b,d, and e.



(c) (5 points) Using the same graph G as in part (b), how many isomorphisms are there from G to itself? Justify your answer.

Solution: The node e must match to itself, because there are no other nodes with degree 8. Similarly, a must match to itself because it's the only node with degree 3.

We have two possible matches for node b: b or d. Once we make this choice, the matches for d, g, and c are all forced by connecting edges.

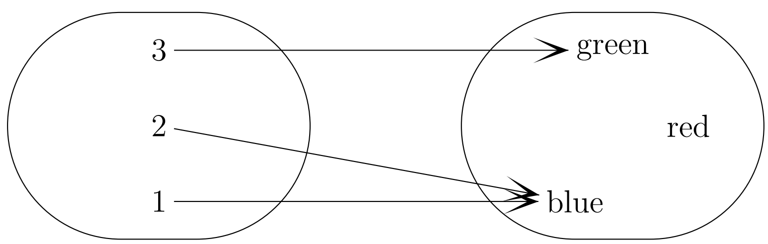
The nodes f, h, and k can all be freely matched permuted. So we have $3! = 6$ choices for how to match all three of them.

So, in total, there are $2 \cdot 6 = 12$ isomorphisms of G to itself.

Problem 3: Functions (13 points)

(a) (5 points) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{\text{red}, \text{green}, \text{blue}\}$. Give an example of a function $f : X \to B$ where $X \subseteq A$ and f is not onto. You must say exactly what's in the set X and use a diagram to show which input values map to which output values. Do not attempt to build a defining equation for f.

Solution: There are many possible solutions. For example, suppose that $X = \{1, 2, 3\}$ and then suppose f looks as in the following diagram. Then f isn't onto because no input produces the output value red.



(b) (8 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by $f(x, y) = (x + y, g(x))$. Prove that f is one-to-one.

Solution: Suppose that (x, y) and (s, t) are two elements of $\mathbb{N}^2$ such that $f(x, y) = f(s, t)$.

Substituting the definition of f into the equation $f(x, y) = f(s, t)$, we get that $(x+y, g(x)) = (x + t, g(s))$. So $x + y = s + t$ and $g(x) = g(s)$.

Because g is one-to-one, $g(x) = g(s)$ implies that $x = s$. So $x + y = s + t$ implies that $x + y = x + t$ so $y = t$.

Since $x = s$ and $y = t$, $(x, y) = (s, t)$, which is what we needed to show.

Problem 4: Tree Induction (13 points)

Suppose that grammar G has these rules: $S \to SS \mid Sb \mid ab \mid b$

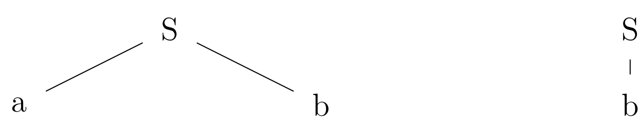
Suppose that the only start symbol is S (i.e. the root must have label S). And that the only terminals are a and b (i.e. a and b are the only possible labels for leaves). Finally, let's use $A(T)$ as shorthand for the number of a's in tree T, and $B(T)$ for the number of b's in T.

Use strong induction to prove that $A(T) \leq B(T)$ for any tree T matching grammar G.

Solution:

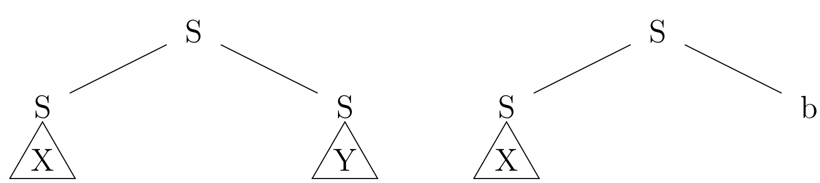
The induction variable is named h and it is the height of the tree.

Base Case(s): The shortest trees for this grammar have height 1. There are two possible trees of this height. Since both trees clearly have at least as many b's as a's, the claim is true for both of them.



Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $A(T) \leq B(T)$ for any tree T of height $< k$ which matches grammar G.

Inductive Step: Suppose that T is a tree of height $k \geq 2$ matching grammar G. The top of T must look like one of the following:



In the lefthand case, the claim holds for the subtrees X and Y by the inductive hypothesis. So $A(X) \leq B(X)$ and $A(Y) \leq B(Y)$. But then $A(T) = A(X) + A(Y) \leq B(X) + B(Y) = B(T)$. So $A(T) \leq B(T)$, which is what we needed to show.

In the righthand case, the claim holds for the subtree X by the inductive hypothesis. So $A(X) \leq B(X)$. But then $A(T) = A(X) \leq B(X) \leq B(X) + 1 = B(T)$. So $A(T) \leq B(T)$, which is what we needed to show.

Problem 5: Induction (15 points)

Let function $f : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by

$$f(1) = 0$$

$$f(2) = 12$$

$$f(n) = 4 \cdot f(n-1) - 3 \cdot f(n-2), \quad \text{for } n \geq 3$$

Use strong induction on n to prove that $f(n) = 2 \cdot 3^n - 6$ for any positive integer n.

Base case(s): For $n = 1$, $f(1) = 0$ and $2 \cdot 3^n - 6 = 2\dot{3} - 6 = 0$. So the claim is true.

For $n = 2$, $f(2) = 12$ and $2 \cdot 3^n - 6 = 2\dot{3}^2 - 6 = 18 - 6 = 12$. So the claim is true.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 2 \cdot 3^n - 6$ for $n = 1, 2, \ldots, k-1$ for some positive integer $k \geq 3$.

Rest of the inductive step: $f(k) = 4 \cdot f(k-1) - 3 \cdot f(k-2)$ by the definition of f.

So $f(k) = 4 \cdot (2 \cdot 3^{k-1} - 6) - 3 \cdot (2 \cdot 3^{k-2} - 6)$ by the inductive hypothesis.

So $f(k) = 8 \cdot 3^{k-1} - 24 - 6 \cdot 3^{k-2} + 18 = 8 \cdot 3^{k-1} - 2 \cdot 3^{k-1} - 6 = 6 \cdot 3^{k-1} - 6 = 2 \cdot 3^k - 6$

So $f(k) = 2 \cdot 3^k - 6$ which is what we needed to show.