CS 173 Spring 2014
Review problems for the second midterm

1. **Counting**

   (a) In our role-playing game, an evil character may be an elf or a troll, it may be red, green, brown, or black, and it may have scales or hair. A good character may be an elf or a human or a lion, it may be green, brown, or blue, and it has hair or fur. How many different character types do we have?

   (b) Suppose we have a 26 character alphabet. How many 6-letter strings start with PRE or end with TH?

   (c) How many different 6-letter strings can I make out of the letters in the word "illini"?

2. **Nested Quantifiers**

   Prove or disprove the statements in (a), (b), and (d). **Hint**: these proofs/disproofs are meant to be very brief.

   (a) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ \text{GCD}(x, y) = 1$

   (b) $\forall x \in \mathbb{Z}^+,\ \exists y \in \mathbb{Z}^+,\ x = y^2$

   (c) Suppose that f is a function from $\mathbb{Z}_6$ to $\mathbb{Z}_8$, and $\exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$. Give a one sentence description of the function f.

   (d) $\exists f : \mathbb{Z}_6 \to \mathbb{Z}_8,\ \exists c \in \mathbb{Z}_8,\ \forall x \in \mathbb{Z}_6,\ f(x) = c$

Function Proofs

(a) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if $f \circ g$ is onto, then g is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

(b) Suppose that $g : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define the function $f : \mathbb{Z} \to \mathbb{Z}^2$ by $f(x) = (x^2, g(x))$. Prove that f is one-to-one.

(c) Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n + 1) = 5f(n) - 6f(n - 1)$

Suppose we're proving that $f(n) = 3^n - 2^n$ for every positive integer n. State the inductive hypothesis and the conclusion of the inductive step.

Induction

Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 1$

$f(1) = 6$

$\forall n \geq 2,\ f(n) = 6f(n-1) - 9f(n-2)$

Use strong induction on n to prove that $\forall n \geq 0,\ f(n) = (1+n)3^n$.

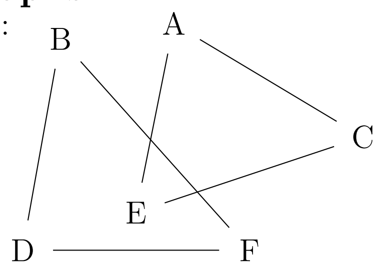
Base case(s):

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

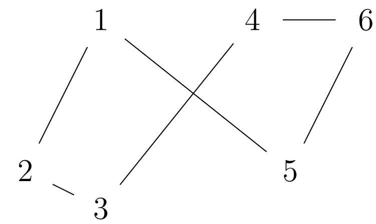
Rest of the inductive step:

Graphs

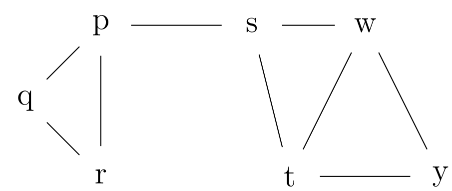
G1:



G2:



G3:



(a) How many connected components does each graph have?

(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.

(c) What is the diameter of G3?

(d) Does G3 contain an Euler circuit? Why or why not?

(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.

(f) How many isomorphisms are there from G3 to G3? Justify your answer or show work.

3. **Recursion Trees** Use a recursion tree to find the closed form expression for the function T defined by

$$T(1) = c$$
$$T(n) = 3T(n/3) + n$$

(a) At level k, how many nodes are there and what value does each contain?

(b) For input value n, what is the level of the leaf nodes?

(c) For any non-leaf level k, what is the sum of values in the nodes?

(d) What is the total value of the leaf nodes?

(e) What is the total value of all nodes, including all levels of the tree?

4. **Tree induction**

A Pioneer tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 5, 17, or 23.
- A node with one child contains the same number as its child.
- A node with two children contains the value $x(y+1)$, where x and y are the values in its children.

Use strong induction to prove that the value in the root of a Pioneer tree is always positive.

The induction variable is named ______ and it is the ____________________ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step: