CS 173, Spring 2014
Midterm 2, B Lecture

NAME:

NETID (e.g. hpotter23, not 314159265):

Circle your discussion:

Th 2 Th 3 Th 4 Th 5 Fri 10 Fri 11 Fri 4

Problem	1	2a	2b	3a	3b	1-3
Possible	12	10	8	10	10	50
Score						

Problem	4	5	6		4-6
Possible	14	18	18		50
Score					

Total out of 100 points

We will be checking photo ID's during the exam. Have your ID handy.
(Forgot your ID? See us at the end of the exam.)

Turn in your exam at the front when you are done.

You have 75 minutes to finish the exam.

INSTRUCTIONS (read carefully)

- There are 6 problems, each on a separate page. Make sure you have a complete exam. The point value of each problem is indicated next to the problem, and on the cover page table.
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- When writing proofs, use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order.
- Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. Partial credit for multiple-choice questions is very rare.
- It is not necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, 3^{17}, and $7!$, unless it is explicitly indicated to completely simplify.
- It is wise to skim all problems and point values first, to best plan your time.
- This is a closed book exam.
  Turn off your cell phone now.
  No notes or electronic devices of any kind are allowed.
  These should be secured in your bag and out of reach during the exam.
- Do all work in the space provided, using the backs of sheets if necessary.
  If your work is on the backside then you must clearly indicate so on the problem.
  See the proctor if you need more paper.
- Please bring any apparent bugs or ambiguity to the attention of the proctors.
- After the midterm is over, you may discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a makeup exam).

Problem 1: Checkbox (12 points)

Check the box that best characterizes each item. (2 points each)

$\forall a \in \mathbb{N}, \quad \exists (b,c) \in \mathbb{Z}^2,$
$(b = a) \ \wedge \ (c = -a)$

true ☑ **false** ☐

$f : \mathbb{Z} \to \mathbb{N}, f(x) = |x|$ is

- **one-to-one but not onto** ☐
- **onto but not one-to-one** ☑
- **neither one-to-one nor onto** ☐
- **bijective** ☐
- **not a valid function** ☐

The diameter of a W_{15} graph is

2 ☑ **3** ☐
14 ☐ **15** ☐

The chromatic number of a graph with a K_5 subgraph is

at least 5 ☑ **at most 5** ☐
exactly 5 ☐ **none of the above** ☐

How many ways can I choose n players from among $n+1$ candidates?

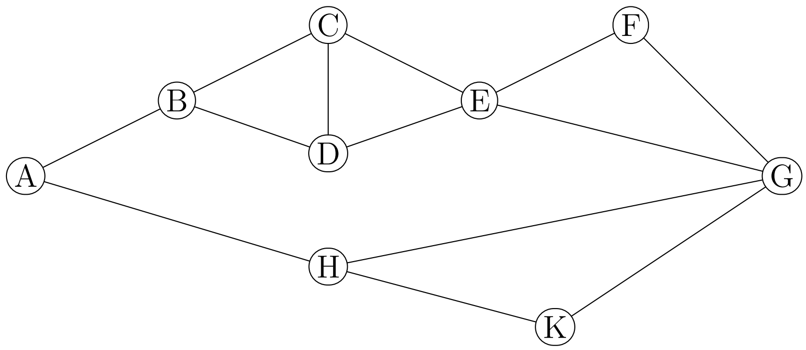
n ☐ $n+1$ ☑
$n!$ ☐ $(n+1)!$ ☐

The diameter of a full and complete binary tree with height h is

h ☐ $h+1$ ☐
$h-1$ ☐ $2h$ ☑

Problem 2: Short answer (18 points)

(a) (10 points) Recall that a path never re-uses a node. How many paths are there from K to C in the following graph? Explain or show work.



Solution: A path from K to C must either go via G or A. A path through G must go through E and cannot go through A (because nodes cannot be re-used). Similarly, a path through A must go through H and B.

There are two paths from K to G (without passing through A and E), two paths from G to E, and three paths from E to C (without passing through G), yielding 12 paths from K to C through G and E.

There are two paths from K to A, without going through B, and three paths from A to C without going through H, yielding 6 paths from K to C through A.

All paths from K to C pass through either A or G, so there are 18 paths total from K to C.

Rubric:

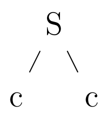
- 3 points for providing at least some valid paths (but incomplete answer, no combinations correctly used)
- 7 points for attempting to solve through combinations of shorter paths but not reaching correct answer
- 10 points for achieving correct answer (through enumeration or more clever methods)

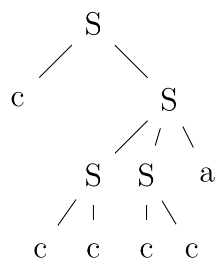
(b) (8 points) Here is a grammar (with start variable S and terminals a and c). Circle the trees that match the grammar.

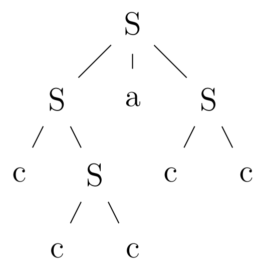
Solution: The first two trees are valid; the others aren't.

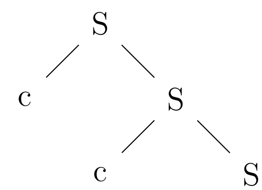
Rubric: 2 points per correct answer.

$$S \rightarrow S\,S\,a \mid c\,S \mid c\,c$$


S
c c

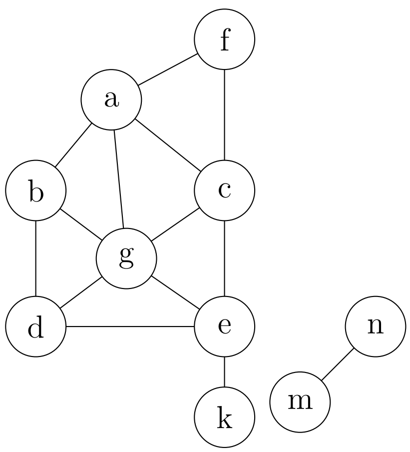

S
c S
S S a
c c c c


S
S a S
c S c c
c c


S
c S
c S

Problem 3: Short Answer (20 points)

(a) (10 points) Answer the questions about the graph on the left. (No need to justify.)



What is its chromatic number? 4 (from W_5 subgraph)

Has an Euler circuit? No (e.g., some nodes of degree 1)

Is it bipartite? No (chromatic number more than 2)

How many connected components? 2

What is the degree of g? 5

(b) (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (f(x) + y, y + 3)$. Prove that g is onto.

Let (a, b) be an arbitrary pair of integers. Let $y = b-3$. Then y is an integer because b is an integer, and $b = y + 3$. Also, note that $f(x) + y = a$ if $f(x) = a - y$. Since $a - y$ is an integer and f is onto, there must be some integer x such that $f(x) = a - y$, so $a = f(x) + y$ for some x. Therefore, there exist some integers (x, y) such that $g(x, y) = (a, b)$, so g is onto by definition of onto. QED.

Rubric:

- Proof attempts to show that there exist input integers to g that map to an arbitrary pair of output integers (+3 points)
- Proof demonstrates why second input integer (b in above solution) has a corresponding input (+2 points)
- Proof demonstrates why first input integer (a in above solution) has a corresponding input using the onto property of f (+3 points)
- Proof is complete and clear (+2 points)

Problem 4: Recursion Tree (14 points)

Suppose that we are building a recursion tree for the function T, defined as follows:

$$T(1) = c \qquad \text{and} \qquad T(n) = nT(n-1) + n$$

(a) How many nodes are there in level 2, i.e. two levels below the root? (3 points)
$n(n-1)$

(b) What is the value in each node in level 2? (2 points)
$n-2$

(c) For any non-leaf level k, with $k > 0$, what is the sum of values in the nodes? (3 points)
$\frac{n!}{(n-k)!}(n-k) = \frac{n!}{(n-k-1)!}$

(d) What is the level of the leaf nodes? (3 points)
$1 = n - k$ so $k = n - 1$

(e) What is the sum of all the values in the leaf nodes? (3 points) $\frac{n!}{(n-n+1)!}c = (n!)c$

Rubric: Partial credit possible for slight errors (e.g., off by one).

Problem 5: Tree Induction (18 points)

Let's define a SM2 tree to be a full and complete binary tree obeying the following rules

- The tree has height at least 1
- The value in an internal node is 0 if the sum of values of its children is even
- The value in an internal node is 1 if the sum of values of its children is odd
- Each leaf can take any integer value

Use strong induction to prove that, for all SM2 trees, the root value is 0 if the sum of all leaf values is even, and the root value is 1 if the sum of all leaf values is odd.

The induction variable is named h and it is the *height* of/in the tree.

Base Case(s): Let T be an SM2 tree of height 1. By definition of SM2, the root has value 1 if the sum of values of its children is even and 0 otherwise. The root's children are the leaves, so the root has value 1 if the leaves sum to an even number and value 0 if the leaves sum to an odd number.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that for all SM2 trees of height of $1, 2, 3, ..., k$ that the root's value is 0 if the leaves sum to an even number and 1 if the leaves sum to an odd number.

Inductive step: I need to show for all SM2 trees of height $k + 1$ that the root's value is 0 if the leaves sum to an even number and 1 if the leaves sum to an odd number. Suppose T is an SM2 tree of height $k + 1$. Because the tree is full and complete, the root must have two children that are the roots of subtrees of height k. I will call the subtrees $C1$ and $C2$ and the values of their roots $x1$ and $x2$.

Case 1: Suppose the leaves of each $C1$ and $C2$ sum to an even number. Then, the sum of all leaves under T is even (even + even is even). Also, by IH, $x1 = 0$ and $x2 = 0$, which sum to 0 (an even number), so the value of T's root is 0.

Case 2: Suppose the leaves of each $C1$ and $C2$ sum to an odd number. Then the sum of all leaves under T is even (odd+odd is even). Also, by IH, $x1 = 1$ and $x2 = 1$, which sum to 2 (an even number), so the root of T has value 0.

Case 3: Suppose the leaves of one child sum to an odd number and the leaves of the other child sums to an even number. Then the total sum of leaves under T is odd (odd+even is odd). Also, by the IH, the values of the children's roots will be 1 and 0, which sums to 1 (an odd number), so the root of T has value 1.

QED.

Rubric:

- Correctly identified induction variable: 2 points
- Correctly specified and justified base cases: 4 points
- Correct specification of inductive hypothesis: 4 points
- Induction step splits into subtrees rooted in the children of the root and uses inductive hypothesis: 4 points
- Clarity and completeness/correctness: 4 points

Write your netID, in case this page gets pulled off:

Problem 6: Induction (18 points)

Let's define a function $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 2 \\ f(n) &= 2f(n-1) + 3f(n-2) + 4 \text{ for } n \geq 2 \end{aligned}$$

Use (strong) induction to prove that $f(n) = 3^n - 1$ for all $n \geq 1$.

Proof by induction on n

Base case(s): Let $n = 0$. $f(n) = 3^0 - 1 = 0$. Let $n = 1$. $f(n) = 3^1 - 1 = 2$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose for $n = 0, 1, 2, ..., k$ with $k \geq 1$ that $f(n) = 3^n - 1$.

Rest of the inductive step: I need to show that $f(k+1) = 3^{k+1} - 1$. By the recursive definition, $f(k+1) = 2f(k) + 3f(k-1) + 4$. By IH, $f(k) = 3^k - 1$ and $f(k-1) = 3^{k-1} - 1$, so $f(k+1) = 2(3^k - 1) + 3(3^{k-1} - 1) + 4 = 2 * 3^k + 3^k - 5 + 4 = 3 * 3^k - 1 = 3^{k+1} - 1$. QED.

Rubric:

- Correctly specified and justified base cases: 4 points
- Correct specification of inductive hypothesis: 4 points
- Induction step uses inductive hypothesis: 4 points
- Clarity and correct algebra: 4 points
- Proof is complete and correct: 2 points