CS 173: Discrete Structures, Spring 2014
Exam 1 Review Solutions

1. **Euclidean algorithm**

   Trace the execution of the Euclidean algorithm for computing GCD on the inputs $a = 837$ and $b = 2015$. That is, give a table showing the values of the main variables (x, y, r) for each pass through the loop. Explicitly indicate what the output value is.

   Solution:

x	y	r
837	2015	837
2015	837	341
837	341	155
341	155	31
155	31	0
31	0	

   Therefore, the algorithm outputs GCD(837, 2015) = 31. Note that the algorithm terminates when y = 0, **not** when r = 0.

2. **Direct Proof Using Congruence mod k**

   In the book, you will find several equivalent ways to define congruence mod k. For this problem, use the following definition: for any integers x and y and any positive integer m, $x \equiv y \pmod{m}$ if there is an integer k such that $x = y + km$.

   Using this definition prove that, for all integers a, b, c, p, q where p and q are positive, if $a \equiv b \pmod{p}$ and $c \equiv b \pmod{q}$ and $q|p$, then $a - 2c \equiv (-b) \pmod{q}$.

   Solution:
   Let a, b, c, p, q be integers, where p and q are positive. Suppose that $a \equiv b \pmod{p}$ and $c \equiv b \pmod{q}$ and $q|p$. By the given definition of congruence, $a = b + pr$ and $c = b + qt$, where r and t are integers. Since $q|p$, we know that $p = qu$, where u is an integer.

   Therefore, by substituting $b + pr$ for a and $b + qt$ for c:

   $$a - 2c = b + pr - 2(b + qt)$$

   By substituting qu for p, we get:

   $$\begin{aligned} a - 2c &= b + qur - 2(b + qt) = b + qur - 2b - 2qt \\ &= (-b) + q(ur - 2t) = (-b) + qw \end{aligned}$$

   where $w = ur - 2t$. By closure, w must be an integer. Therefore, by the definition given for congruence, $a - 2c \equiv (-b) \pmod{q}$.

3. **Equivalence classes**

Let $A = \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} - \{(0,0)\}$, i.e. pairs of non-negative reals in which no more than one of the two numbers is zero.

Consider the equivalence relation $\sim$ on A defined by

$$(x,y) \sim (p,q) \quad \text{iff} \quad (xy)(p+q) = (pq)(x+y)$$

(a) List four elements of $[(3,1)]$. Hint: what equation do you get if you set (x,y) to $(3,1)$ and $q = 2p$?

(b) Give two other distinct equivalence classes that are not equal to $[(3,1)]$.

(c) Describe the members of $[(0,4)]$.

Solutions:

(a) $(3,1)$, $(1,3)$, $(\frac{9}{8}, \frac{9}{4})$, $(\frac{9}{4}, \frac{9}{8})$. You can find a range of other elements by setting q to other multiples of p.

(b) For example, $[(3,2)]$, $[(3,4)]$

(c) All pairs of the form $(0,y)$ or $(x,0)$.
If $(x,y) = (0,4)$, then the equation $(xy)(p+q) = (pq)(x+y)$ reduces to $0(p+q) = (pq)4$. So this means either p or q must also be zero and, then, it doesn't matter what value we give to the other.

4. **Subset proof**

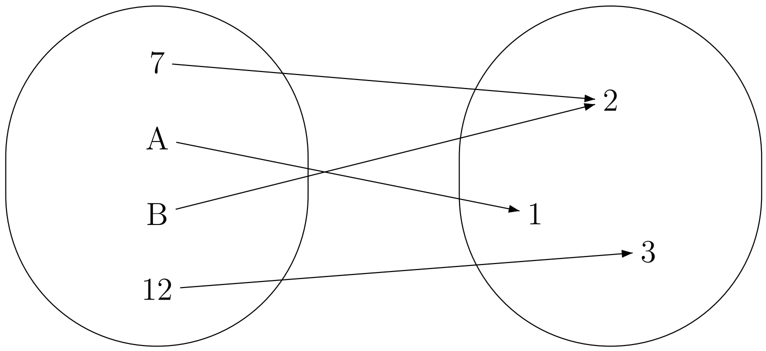
Suppose that A, B and C are sets. Recall the definition of $X \subseteq Y$: for every p, if $p \in X$, then $p \in Y$. Prove that if $A \subseteq B$ then $A \cap C \subseteq B \cap C$. Briefly justify the key steps in your proof.

Solution: Suppose that $p \in A \cap C$. Then $p \in A$ and $p \in C$, by the definition of intersection. Since $p \in A$ and $A \subseteq B$, $p \in B$ (definition of subset). So $p \in B$ and $p \in C$, which implies that $p \in B \cap C$ (definition of intersection).

5. **Functions**

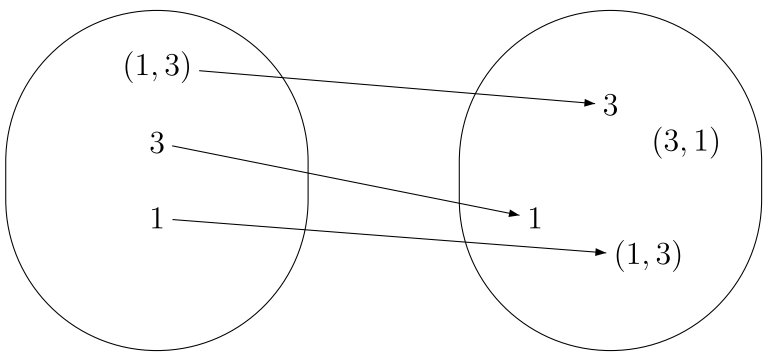
For each of the following functions determine if it is onto or not onto. Briefly, but clearly, justify your answers. (A full formal proof is not required.)

(a) The function f given by the following diagram where the left bubble represents the domain and the right the codomain:



Solution: The function f is onto because every output has at least one corresponding input that the function maps to it.

(b) The function g given by the following diagram:



Solution: The function g is not onto because the codomain element $(3, 1)$ has no corresponding input that maps to it.

(c) $h : \mathbb{Z} \to \mathbb{Z}$ such that $h(x) = 3\lceil \frac{x}{3} \rceil$

Solution: The function h is not onto because 1 is not in the image of the function. If it were, then $1 = 3\lceil \frac{x}{3} \rceil$ which is impossible because $\lceil \frac{x}{3} \rceil$ is an integer.

(d) $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(x, y) = x$

Solution: The function k is onto. Pick any codomain element $x \in \mathbb{R}$. Consider $(x, 0) \in \mathbb{R} \times \mathbb{R}$. Notice that $k(x, 0) = x$, so x has a pre-image.

6. **One-to-one**

Which of these functions are one-to-one? Briefly justify your answers.

(a) $h : [0, 1] \to \mathbb{R}^2$ such that $h(\lambda) = \lambda(2, 2) + (1 - \lambda)(1, 3)$ where you use the following formula to multiply a real number a by a 2D point (x, y):

$$a(x, y) = (ax, ay)$$

Solution

h is one-to-one. In $\mathbb{R}^2$, h describes the strictly increasing line segment between the points (2,2) and (1,3).

(b) $f : \mathbb{N}^2 \to \mathbb{N}$ such that $f(x, y) = 4^x 3^y$

Solution

f is one-to-one. The image of f is the set of positive integers that have only 2 and 3 as prime factors and the prime factorization of any integer is unique.

(c) $k : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ such that $k(x, y) = (1 - x^2) \left\lfloor \frac{y}{3} \right\rfloor$

Solution

k is not one-to-one. (0,0) and (1,0) are both mapped to 0.