CS 173, Fall 2013, Midterm 1 "Potter" solutions

Problem 1: Multiple choice (10 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$\lfloor x \rfloor < \lceil x \rceil$	true for any real number x	☐
	false for any real number x	☐
	true for some real numbers x	☑ √

	True	False
For any integer x, if $x \geq 8$, then $x^2 \geq 36$.	☑ √	☐
$p \rightarrow \neg q \equiv q \rightarrow \neg p$	☑ √	☐

$\lvert A \cup B \rvert = \lvert B \rvert + \lvert A \rvert$	True for any sets A and B	☐
	False for any sets A and B	☐
	True for some sets A and B	☑ √

$\emptyset \in A$	true for any set A	☐
	false for any set A	☐
	true for some sets A	☑ √

Problem 2: Multiple choice (12 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$-3 \equiv 7 \pmod{10}$	True	☑ √	False	☐
For any positive integers a and b, $\gcd(a, b) = \gcd(b \bmod a, a)$	True	☑ √	False	☐
For any positive integers p and q, if $\text{lcm}(p, q) = pq$, then p and q are relatively prime.	True	☑ √	False	☐
A relation cannot be both symmetric and antisymmetric.	True	☐	False	☑ √

A linear order is a special type of partial order. What extra property does it have?

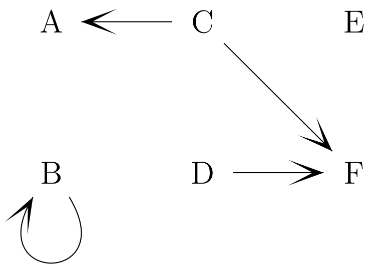
- all elements are reflexive ☐
- all pairs of elements are comparable ☑ √
- it is strict ☐
- it is homogeneous ☐

If $f : \mathbb{Z} \to \mathbb{R}$ is a function such that $f(x) = 2x$ then $\mathbb{R}$ is

- the domain of f ☐
- the co-domain of f ☑ √
- the image of f ☐

Problem 3: Short answer (16 points)

(a) (10 points) Check all boxes that correctly characterize this relation on the set $\{A, B, C, D, E, F\}$

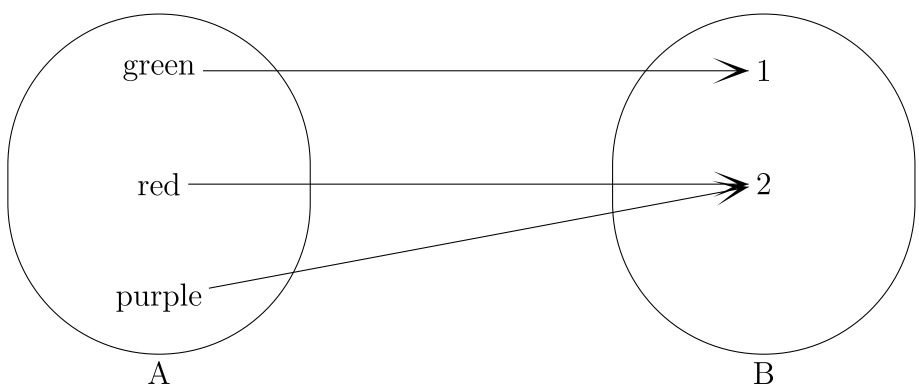


Reflexive: ☐ Irreflexive: ☐

Symmetric: ☐ Antisymmetric: ☑ √

Transitive: ☑ √

(b) (6 points) The following picture shows the contents of a set A. Complete it to make an example of a function from A to B that is onto but not one-to-one. That is, add elements to set B and draw arrows between the two sets showing how input values map to output values. The values in B should be integers.



Problem 4: Short answer (12 points)

(a) (6 points) Let R be the relation on the real numbers such that xRy if and only if $\lfloor x/2 \rfloor = \lfloor y/2 \rfloor$. Describe which values are in [17]. (Do not simply wrap the definition in set-builder notation: make it clear that you understand what's in this set.)

Solution: $\lfloor 17/2 \rfloor = 8$. In order for $\lfloor y/2 \rfloor$ to be equal to 8, y must be at least 16 and smaller than 18. So [13] contains all real numbers between 16 and 18, including 16 but not 18. In shorthand: $[17] = [16, 18)$.

(b) (6 points) In $\mathbb{Z}_{11}$, find the value of $[8]^{22}$. You must show your work, keeping all numbers in your calculations small. **You may not use a calculator.** You must express your final answer as $[n]$, where $0 \leq n \leq 10$.

Solution: One good method involves repeatedly squaring 8, mod 11. That is

$$[8]^2 = [64] = [9]$$
$$[8]^4 = [81] = [4]$$
$$[8]^8 = [16] = [5]$$
$$[8]^{16} = [25] = [3]$$

Then

$$[8]^{22}x = [8]^{16+4+2} = [8]^{16} \times [8]^4 \times [8]^2 = [3] \times [4] \times [9] = [12] \times [9] = [1] \times [9] = [9]$$

Problem 5: Short answer (18 points)

(a) (6 points) Suppose that R is a relation on a set A. Using precise mathematical words and notation, define what it means for R to be antisymmetric.

Solution: For every x and y in A, if xRy and $x \neq y$, then $y \not R x$. Or: for every x and y in A, if xRy and yRx, then $x = y$.

(b) (6 points) Is the following claim true? Informally explain why it is, or give a concrete counter-example showing that it is not.

Claim: For any positive integers a, b, k, j, if $a \equiv b \pmod{k}$ and $k|j$, then $a \equiv b \pmod{j}$.

Solution: This is not true. Suppose we let $a = 2$, $b = 7$, $k = 5$ and $j = 10$. Then $2 \equiv 7 \pmod{5}$ and $5 \mid q$, but it's not the case that $2 \equiv 7 \pmod{10}$.

(c) (6 points) State the negation of the following claim. Your answer should be in words, with all negations (e.g. "not") on individual predicates.

For every cat k, if k is orange, then k is large or k is not friendly.

Solution: There is a cat k such that k is orange but k is not large and k is friendly.

Problem 6: Relation Proof (16 points)

Let $A = \mathbb{R}^+ \times \mathbb{R}^+$, i.e A is the set of pairs of positive real numbers. Suppose that T is the relation on A such that $(a, b)T(p, q)$ if and only if $bp \geq aq$. Prove that T is transitive. Hint: it's ok to use division.

Solution: Let (a, b), (m, n) and (p, q) be elements of A and suppose that $(a, b)T(m, n)$ and $(m, n)T(p, q)$.

By the definition of T, this means that $bm \geq an$ an also $np \geq mq$.

We can divide by b, n, and q because all our numbers are positive. So then $\frac{m}{n} \geq \frac{a}{b}$ and $\frac{p}{q} \geq \frac{m}{n}$. Combining these two inequalities, we find that $\frac{p}{q} \geq \frac{a}{b}$. So then $bp \geq aq$.

Since $bp \geq aq$, by the definition of T tells us that $(a, b)T(p, q)$, which is what we needed to show.

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Problem 7: Set theory proof (16 points)

$$A = \{(p, q) \in \mathbb{Z}^2 \ \mid 2pq + 6q - 5p - 15 \geq 0\}$$
$$B = \{(s, t) \in \mathbb{Z}^2 \ \mid s \geq 0\}$$
$$C = \{(x, y) \in \mathbb{Z}^2 \ \mid y \geq 0\}$$

Prove that $(A \cap B) \subseteq C$.

Use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order. You must use the method of selecting an element from the smaller set and showing that it belongs to the larger set.

Solution: Let A, B and C be the sets described above. Suppose that (m, n) is an element of $A \cap B$. We need to show that (m, n) is in C.

Notice that $2mn + 6n - 5m - 15 = (2n - 5)(m + 3)$.

Since (m, n) is in $A \cap B$, (m, n) is in A and also (m, n) is in B.

Since (m, n) is in B, the definition of B implies that $m \geq 0$. So $(m + 3) \geq 0$.

By the definition of A, (m, n) being in A means that $2mn+6n-5m-15 \geq 0$. But $2mn+6n-5m-15 = (2n - 5)(m + 3)$ and we know that $(m + 3) \geq 0$ is non-negative. This implies that $(2n - 5) \geq 0$. But then $2n \geq 5$. So $n \geq \frac{5}{2} \geq 0$.

Since $n \geq 0$, the definition of C tells us that (m, n) is in C, which is what we needed to show.