CS 173, Final Exam
15 December 2012

Fill in your name, netid (eg: alincoln16), and circle your lecture and discussion section below. Also write your name or netid on the last page (which sometimes gets pulled off).

Printed Name:

NetID:

Circle your Discussion Section		
Thr	2	Muntasir
Thr	3	Muntasir
Thr	4	Maxie
Thr	5	Maxie
Fri	9	Estelle
Fri	10	Estelle
Proficiency, B lecture, etc		

Taking as a proficiency?
Write your UIN and any prior CS classes

Scores:		
1		/12
2		/12
3		/13
4		/16
5		/12
6		/11
7		/15
8		/9

Totals:		
1-4		/53
5-8		/47
1-8		/100

INSTRUCTIONS: please read carefully

- There are 8 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem, and on the cover page table. It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- For proofs, be sure to use good mathematical style, with the steps in logical order.
- Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. Partial credit for multiple-choice questions is very rare.
- It is not necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, 3^{17}, and $7!$, unless the problem explicitly says you must do so.
- This is a closed book exam.
  Turn off your cell phone now.
  No notes or electronic devices of any kind are allowed.
  These should be secured in your bag and out of reach during the exam.
- Do all work in the space provided, using the backs of sheets if necessary.
  If your work is on the back side then you must clearly indicate so on the problem.
  Ask a proctor if you need more paper.
- Please bring any apparent bugs or ambiguity to the attention of the proctors.
- We expect most people to finish the exam in 2 hours, but you may take up to the full 3 hours.
- When you are finished, show your ID to the proctors and put your exam in the appropriate marked pile at the front.
- Because of conflict exams (including last-minute emergencies) do not discuss the contents of the exam with other students until the end of the final exam period.

Problem 1: Multiple choice (12 points)

Check the box that best characterizes each item.

If G is a planar graph, then the chromatic number of G is ≤ 6

true ☐ false ☐

Problems in NP need exponential time

proven true ☐ proven false ☐

not known for sure ☐

The running time of mergesort

$O(n)$ ☐ $O(n \log n)$ ☐

$O(n^2)$ ☐ $O(2^n)$ ☐

The complete graph K_5 is planar

true ☐

false ☐ it depends on how you draw K_5 ☐

If $f : \mathbb{R} \rightarrow \mathbb{P}(\mathbb{Z})$ then $f(17)$ is

an integer ☐ a set of integers ☐

one or more integers ☐ a power set ☐

$n!$ is $O(2^n)$

true ☐ false ☐

Problem 2: More multiple choice (12 points)

Check the box that best characterizes each item.

The set of all finite length strings of decimal digits is

finite ☐ countably infinite ☐

uncountable ☐

The set of all functions $f : \mathbb{N} \rightarrow \{a, b, c, d, e, f\}$

finite ☐ countably infinite ☐

uncountable ☐

A partition of a set A contains $\emptyset$

always ☐ sometimes ☐

never ☐

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (zero or more of each variety).

$\binom{17}{5}$ ☐ $\binom{20}{4}$ ☐ $\binom{20}{3}$ ☐

$\binom{17}{4}$ ☐ $\binom{21}{4}$ ☐ $\frac{17!}{4!}$ ☐

$T(1) = d$
$T(n) = T(n/2) + c$

$O(\log n)$ ☐ $O(n)$ ☐

$O(n \log n)$ ☐ $O(n^2)$ ☐

The number of leaves in a binary tree of height h

$\leq 2^h$ ☐ $\geq 2^h$ ☐

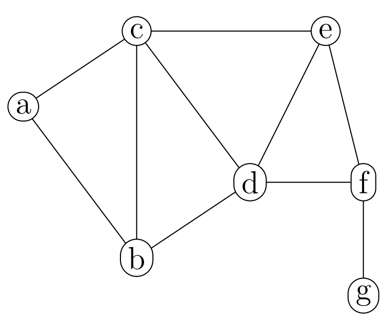
$= 2^h$ ☐

Problem 3: Short answer (13 points)

(a) (7 points) Recall that a phone lattice is a type of state diagram, i.e. a directed graph where each node represents a state of a system. A phone lattice has exactly one letter on each edge. Each path from the start node to a final/end node represents a word. Draw a phone lattice representing exactly the following set of words, using a single start node and no more than 9 nodes total.

```
bat, back, sat, sack
boo, booo, boooo, ...    [i.e. b followed by two or more o's]
moo, mooo, moooo, ...    [i.e. m followed by two or more o's]
```

(b) (6 points) Graph G with set of nodes V is shown below. Recall that $deg(n)$ is the degree of node n. Let's define $f : \mathbb{N} \rightarrow \mathbb{P}(V)$ by $f(k) = \{n \in V : \deg(n) = k\}$. Fill in the selected output values.



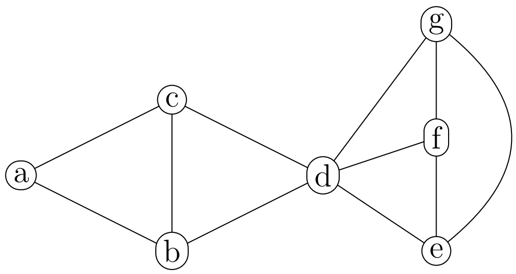
$f(4) =$

$f(1) =$

$f(7) =$

Problem 4: Graph short answer (16 points)

Suppose that G is the graph shown below.



(a) (2 points) Number of faces in G is ...

(b) (2 points) Number of connected components in G is ...

(c) (6 points) What is the chromatic number of G? Justify your answer.

(c) (6 points) How many isomorphisms are there from G to itself. Justify your answer.

Problem 5: Short answer (12 points)

(a) (7 points) Use proof by contradiction to show that $\sqrt{2} + \sqrt{3} \leq 4$.

(b) (5 points) Suppose that $f : A \rightarrow B$ is a function. Define what it means for f to be one-to-one. Make sure your definition is completely precise. Use quantifiers rather than words like "unique."

Problem 6: Algorithm analysis (11 points)

Here is the pseudocode for a function Frog that finds the diameter of a set of 2D points. The function $d(p, q)$ returns the straight-line distance between two points p and q.

```
01 Frog(p_1, ..., p_n : list of n 2D points, n ≥ 3)
02         if (n = 3)
03                 return the largest of d(p_1, p_2), d(p_1, p_3), and d(p_2, p_3)
04         else
05                 x = Frog(p_2, p_3, p_4, ..., p_n)        \\ i.e. remove p_1
06                 y = Frog(p_1, p_3, p_4, ..., p_n)        \\ i.e. remove p_2
07                 z = Frog(p_1, p_2, p_4, ..., p_n)        \\ i.e. remove p_3
08                 return the largest of x, y, and z
```

(a) (4 points) Suppose $T(n)$ is the running time of Frog on an input array of length n. Give a recursive definition of $T(n)$. Assume that setting up the recursive calls in lines 5-7 takes constant time.

(b) (3 points) What is the height of the recursion tree for $T(n)$?

(c) (2 points) How many leaves are in the recursion tree for $T(n)$?

(d) (2 points) What is the big-Theta running time of Frog?

Problem 7: Induction (15 points)

Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 7 \\ f(n) &= 7f(n-1) - 12f(n-2), \qquad \forall n > 2 \end{aligned}$$

Use induction to prove that $f(n) = 4^n - 3^n$ for all $n \in \mathbb{Z}^+$.

Base case(s): [Be careful about logical order of assertions.]

Inductive Hypothesis: [Be specific. Do not refer to "the claim".]

Inductive Step:

Write your netID, in case this page gets pulled off:

Problem 8: Tree induction (9 points)

A wobbly tree is a full binary tree (i.e. each node has 0 or 2 children) whose nodes are labelled with natural numbers such that

- If v is a leaf node, then v has label 15, 30, 35, 40, or 70
- If v has two children with labels x and y, then v has label $\gcd(x, y)$, i.e. the greatest common divisor of x and y. E.g. if the children have labels 30 and 40, the parent has label 10.

Finish the following inductive proof that the root node of every wobbly tree is divisible by 5. Assume we are all familiar with basic facts about divisibility, so you don't need to go all the way back to the definition. Focus on convincing us that you understand tree induction.

Proof: by induction on h, which is the height of the tree.

Base cases: A wobbly tree of height 0 consists of a single node which is both the root and a leaf. So it must have label 15, 30, 35, 40, or 70, all of which are divisible by 5.

Inductive Hypothesis: [Be specific. Do not refer to "the claim".]

Inductive Step: