CS 173, Spring 2013, A Lecture
Midterm 1 Solutions

Problem 1: Multiple choice (15 points)

Check the most appropriate box for each statement. Check only one box per statement. If you change your answer, make sure it's easy to tell which box is your final selection.

$\forall x \in \mathbb{Z},\ x^2 < -3 \ \rightarrow\ x > 10$

true [√] undefined truth value [ ]

false [ ]

$p \rightarrow \neg q \ \equiv\ \neg q \vee \neg p$

True [√] False [ ]

$\gcd(a, 0)$

a [√] 0 [ ]

1 [ ] undefined [ ]

$-7 \equiv 3 \pmod 5$

true [√] false [ ]

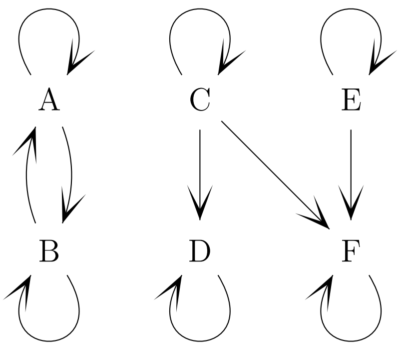
lcm(6, 10)

6 [ ] 10 [ ]

30 [√] 60 [ ]

Problem 2: Short answer (15 points)

(a) (10 points) Check all boxes that correctly characterize this relation on the set $\{A, B, C, D, E, F\}$

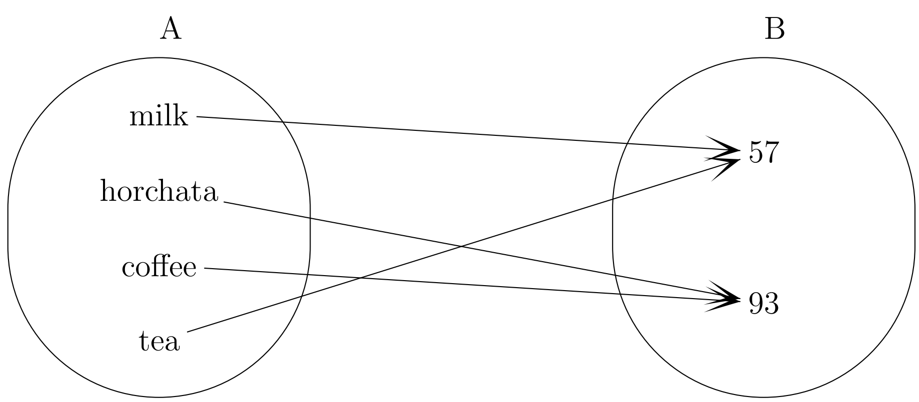


Reflexive: [√] Irreflexive: [ ]

Symmetric: [ ] Antisymmetric: [ ]

Transitive: [√]

(b) (5 points) The following picture shows the contents of a set A. Complete it to make an example of a function from A to B that is onto but not one-to-one. To complete the picture, add elements to set B (represent each element in B with an integer) and draw arrows showing how input values map to output values.



Problem 3: Short answer (12 points)

(a) (6 points) Let $f : \mathbb{R} \to \mathbb{R}, f(x) = (x-1)^2$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 3x$. Write the expression for $(f \circ g)(x)$, and compute $(f \circ g)(3)$. Show your work.

Solution: $(f \circ g)(x) = f(g(x)) = (3x-1)^2$

$(f \circ g)(3) = f(9) = 64$

(b) (6 points) Suppose we have the following sets:

$$\begin{aligned} A &= \{\text{green}, \text{red}\} \\ B &= \{3, 8\} \\ C &= \{4, 8\} \end{aligned}$$

$A \times (B \cup C) = \{(\text{green}, 3), (\text{green}, 4), (\text{green}, 8),$
$(\text{red}, 3), (\text{red}, 4), (\text{red}, 8)\}$

$\{\text{camel}\} \times (B \cap C) = \{(\text{camel}, 8)\}$

$A \cap C = \emptyset$

Problem 4: Short answer (14 points)

(a) (6 points) State the negation of the following claim. Your answer should be in words, with all negations (e.g. "not") on individual predicates.

For every dinosaur d, if d is huge, then d is an adult or d is a sauropod.

Solution: There is a dinosaur d such that d is huge but/and d is not an adult and d is not a sauropod.

(b) (8 points) In $\mathbb{Z}_{13}$, find the value of $[7]^{19}$. You must show your work, keeping all numbers in your calculations small. **You may not use a calculator.** You must express your final answer as $[n]$, where $0 \leq n \leq 12$.

Solution:

$[7]^2 = [49] = [10]$

$[7]^4 = [10]^2 = [100] = [9]$

$[7]^8 = [9]^2 = [81] = [3]$

$[7]^{16} = [3]^2 = [9]$

So $[7]^{19} = [7]^{16}[7]^2[7] = [9][10][7] = [90][7] = [-1][7] = [-7] = [6]$

Problem 5: Relations (12 points)

(a) (6 points) Suppose that R is a partial order on a set A. What additional property is required for R to be a linear order (aka total order)? Give specific details of the property, not just its name.

Solution: All values in A must be comparable. That is, for every x and y in A, either xRy or xRy must be true.

(b) (6 points) Suppose that R is the relation on the set of integers such that pRq if and only if $p \leq q+3$. Is R transitive? Informally explain why it is, or give a concrete counter-example showing that it is not.

Solution: No, it isn't true. Consider the values 6, 3, and 0. Then $6R3$ and $3R0$ but it's not the case that $6R0$.

Problem 6: Proof (15 points)

Recall that $[a, b]$ is a closed interval of the real line. You can assume that $a \leq b$ for any closed interval. Let I be the set containing all closed intervals $[a, b]$. Finally, the relation B on I is defined by

[a,b]B[p,q] if and only if $b \leq p$

There are two equivalent definitions for what it means for R defined on set A to be antisymmetric:

Definition 1: $\forall x, y \in A$, if $x \neq y$ and xRy, then $y \not R x$
Definition 2: $\forall x, y \in A$, if xRy and yRx, then $x = y$

Prove that B is antisymmetric using one of these definitions.

Solution: Let $[c, d]$ and $[e, f]$ be closed intervals and suppose that $[c, d]B[e, f]$ and also $[e, f]B[c, d]$.

By the definition of the relation B, this means that $d \leq e$ and $f \leq c$. We also know, from the definition of closed interval, that $c \leq d$ and $e \leq f$.

Combining these inequalities, we get $f \leq c \leq d \leq e \leq f$. This means that c, d, and e must all be equal to f. So, in particular, $c = e$ and $d = f$. So $[c, d] = [e, f]$, which is what we needed to show.

Problem 7: Proof (17 points)

$$A = \{(p, q) \in \mathbb{Z}^2 \mid p \equiv 3 \pmod{7} \text{ and } q \equiv 4 \pmod{7}\}$$
$$B = \{(s, t) \in \mathbb{Z}^2 \mid st \equiv 5 \pmod{7}\}$$

Congruence mod k can be defined as follows: if a, b, k are integers, k positive, then $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n. Using this definition, prove that $A \subseteq B$.

Use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order. Use only the given definition and algebra, not any facts you might know about modular arithmetic.

Solution: Suppose that (p, q) is an element of A.

By the definition of A, this means that p and q are integers such that $p \equiv 3 \pmod{7}$ and $q \equiv 4 \pmod{7}\}$.

Using the given definition of congruence mod k, this means that $p = 3 + 7k$ and $q = 4 + 7n$ for some integers k and n. So we have

$$pq = (3 + 7k)(4 + 7n) = 12 + 7(4k + 3n + 7kn) = 5 + 7(1 + 4k + 3n + 7kn)$$

Since k and n are integers, so is $1 + 4k + 3n + 7kn$. Let's call it m. Then $pq = 5 + 7m$, where m is an integer. So $pq \equiv 5 \pmod{7}$.

Since (p, q) is a pair of integers such that $pq \equiv 5 \pmod{7}$, (p, q) is in the set B, by the definition of B.