Cardinality

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This is a half lecture due to the makeup quiz. It discussed cardinality, an interesting topic but which doesn't have an obvious fixed place in the syllabus. It's covered at the very end of section 2.4 in Rosen.

1 The rationals and the reals

You're familiar with three basic sets of numbers: the integers, the rationals, and the reals. The integers are obviously discrete, in that there's a big gap between successive pairs of integers.

To a first approximation, the rational numbers and the real numbers seem pretty similar. The rationals are dense in the reals: if I pick any real number x and a distance δ, there is always a rational number within distance δ of x. Between any two real numbers, there is always a rational number.

We know that the reals and the rationals are different sets, because we've shown that a few special numbers are not rational, e.g. π and $\sqrt{2}$. However, these irrational numbers seem like isolated cases. In fact, this intuition is entirely wrong: the vast majority of real numbers are irrational and the rationals are quite a small subset of the reals.

2 Completeness

One big difference between the two sets is that the reals have a so-called "completeness" property. It states that any subset of the reals with an upper bound has a smallest upper bound. (And similarly for lower bounds.) So if I have a sequence of reals that converges, the limit it converges to is also a real number. This isn't true for the rationals. We can make a series of rational numbers that converge π (for example) such as

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, 3.1415926, 3.14159265$$

But there is no rational number equal to π.

In fact, the reals are set up precisely to make completeness work. One way to construct the reals is to construct all convergent sequences of rationals and add new points to represent the limits of these sequences. Most of the machinery of calculus depends on the existence of these extra points.

3 Cardinality

Furthermore, although the rationals and the reals both contain infinitely many points, we can show that the reals have "more" points. To do this, we need to definite what it means for two sets to have the same cardinality, i.e. the same mathematical size.

> Definition: Two sets A and B have the same cardinality if and only if there is a bijection from A to B.

Finite sets have the same cardinality exactly when they have the same number of elements in the usual sense. But not all infinite sets have the same cardinality.

> An infinite set A is *countable* or *countably infinite* if there is a bijection from $\mathbb{Z}^+$ onto A.

The full set of integers is countable, because we can map the natural numbers onto the integers using the function f where $f(n) = \frac{n}{2}$ when n is even and $f(n) = \frac{-n+1}{2}$ when n is odd.

The positive rationals are countable because we can put them into an ordered list. [show picture, which is on Rosen p. 159]. It's not hard to extend this idea to also include zero and the negative rationals.

However, we can show that the reals are not countable. Specifically, we'll show that there's no bijection from the positive integers onto the real interval $[0, 1]$ using a construction called "diagonalization" developed by Georg Cantor.

If the numbers in $[0, 1]$ were countable, we could put them into a list a_1, a_2, and so forth. Let's write out a table of the decimal expansions of the numbers on this list. [see picture p. 160 of Rosen.] Now, examine the digits along the diagonal of this table: a_{11}, a_{22}, etc. Suppose we construct a new number b whose kth digit b_k is 4 when a_{kk} is 5, and 5 otherwise. Then b won't match any of the numbers in our table, so our table wasn't a complete list of all the numbers in $[0, 1]$.

So, the reals are larger than the integers.

4 Uncomputability

A practical consequence of this difference in size is that there are mathematical functions that can't be computed by any program. First, consider the functions $f : \mathbb{N} \to D$ (where D is the decimal digits). Each function corresponds to the decimal expansion of some real number in $[0, 1]$. So even this limited set of functions is uncountable.

However, suppose we fix an alphabet for writing our programs (e.g. 8-bit ASCII). Since each individual program is finite in length, we can put all possible programs into a (very long) ordered list. For any fixed character length k, there are only a finite set of possible programs. So, we can write down all programs by first first writing down all the 1-character programs, then all the 2-character programs, and so forth. In other words, there's a bijection between the integers and the total set of programs.

But this means that the number of functions is uncountable, whereas the number of programs is only countably infinite. So there must be mathematical functions that we can't compute with any (finite-length) program.