Planar Graphs II

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This lecture continues the discussion of planar graphs (section 9.7 of Rosen).

1 Announcements

Makeup quiz last day of classes (at the start of class).

Your room for the final exam (Friday the 7th, 7-10pm) is based on the first letter of your last name:

- A-H Roger Adams Lab 116
- I-W Business Instructional Facility (515 E. Gregory opposite Armory) 1001
- X-Z Roger Adams Lab 116

The conflict exam (Monday the 10th, 1:30-4:30pm) is in 269 Everett Lab.

2 A corollary of Euler's formula

Suppose G is a connected simple planar graph, with v vertices, e edges, and f faces, where $v \geq 3$. Then $e \leq 3v - 6$.

Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 3. So we have $2e \geq 3f$. Then $\frac{2}{3}e \geq f$.

Euler's formula says that $v-e+f=2$, so $f=e-v+2$. Combining this with $\frac{2}{3}e \geq f$, we get

$$e - v + 2 \leq \frac{2}{3}e$$

So $\frac{e}{3} - v + 2 \leq 0$. So $\frac{e}{3} \leq v - 2$. Therefore $e \leq 3v - 6$.

We can also use this formula to show that the graph K_5 isn't planar. K_5 has five vertices and 10 edges. This isn't consistent with the formula $e \leq 3v - 6$. Unfortunately, this trick doesn't work for $K_{3,3}$, which isn't planar but satisfies the equation (with 6 vertices and 9 edges).

3 Another corollary

In a similar way, we can show that if G is a connected planar simple graph with e edges and v vertices, with $v \geq 3$, and if G has no circuits of length 3, then $e \leq 2v - 4$.

Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 4 because we have no circuits of length 3. So we have $2e \geq 4f$. Then $\frac{1}{2}e \geq f$.

Euler's formula says that $v-e+f=2$, so $e-v+2=f$. Combining this with $\frac{1}{2}e \geq f$, we get

$$e - v + 2 \leq \frac{1}{2}e$$

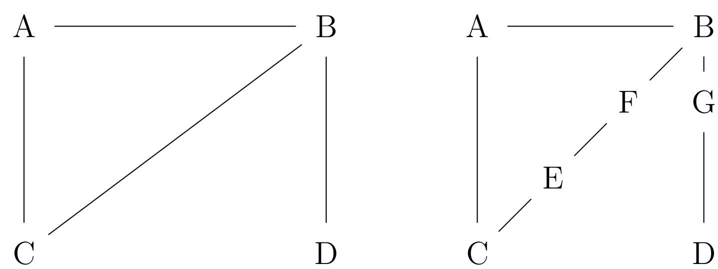
So $\frac{e}{2} - v + 2 \leq 0$. So $\frac{e}{2} \leq v - 2$. Therefore $e \leq 2v - 4$.

We can use this formula to show that $K_{3,3}$ isn't planar.

4 Kuratowski's Theorem

The two example non-planar graphs $K3, 3$ and K_5 weren't picked randomly. It turns out that any non-planar graph must contain a copy of one of these two graphs. Or, sort-of. The copy of $K3, 3$ and K_5 doesn't actually have exactly the literal vertex and edge structure of one of those graphs (i.e. be isomorphic). We need to define a looser notion of graph equivalence, called *homeomorphism*.

A graph G is a *subdivision* of another graph F if G is just like F except that you've divided up some of F's edges by adding vertices in the middle of them. For example, in the following picture, the righthand graph is a subdivision of the lefthand graph.



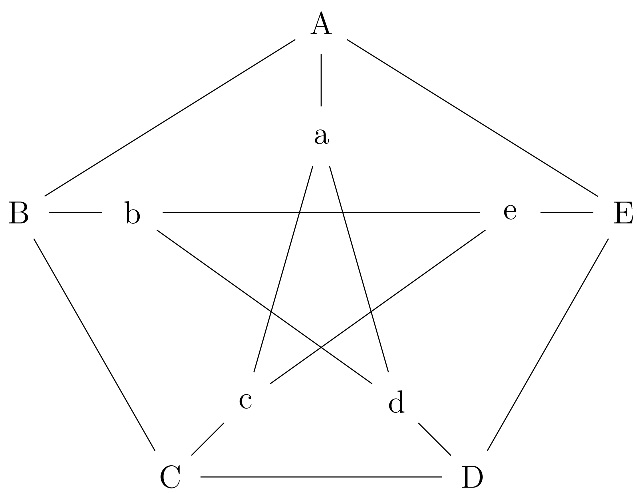
Two graphs are *homeomorphic* if one is a subdivision of another, or they are both subdivisions of some third graph. Graph homeomorphism is a special case of a very general concept from topology: two objects are homeomorphic if you can set up a bijection between their points which is continuous in both directions. For surfaces (e.g. a rubber ball), it means that you can stretch or deform parts of the surface, but not cut holes in it or paste bits of it together.

We can now state our theorem precisely.

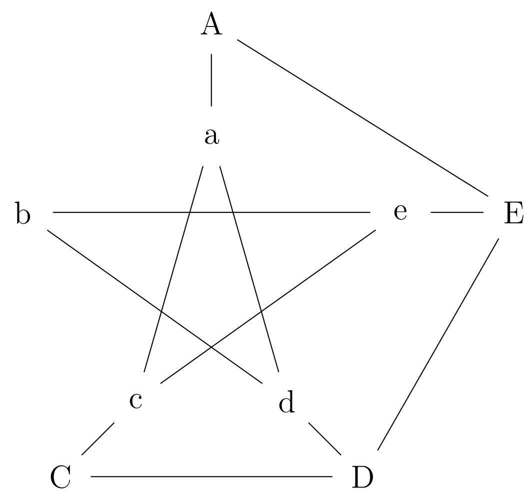
Claim 1 *Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to* $K_{3,3}$ *or* K_5*.*

This was proved in 1930 by Kazimierz Kuratowski, and the proof is apparently somewhat difficult. So we'll just see how to apply it.

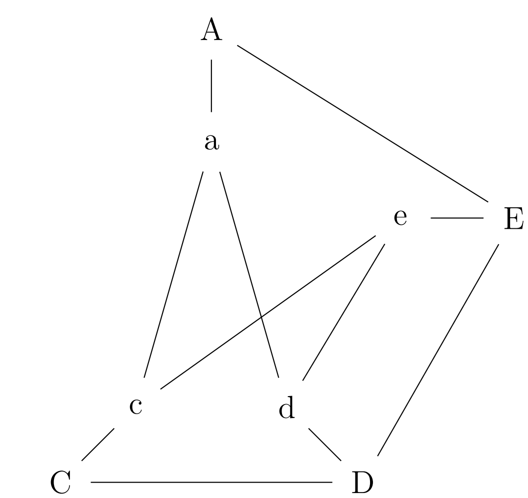
For example, here's a graph known as the Petersen graph (after a Danish mathematician named Julius Petersen).



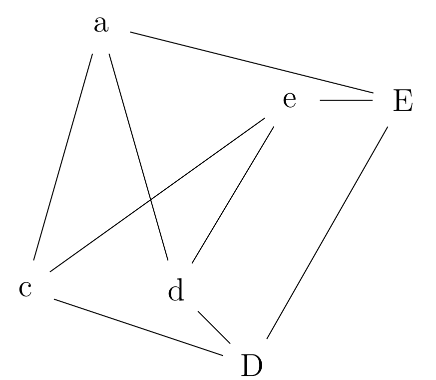
This isn't planar. The offending subgraph is the whole graph, except for the node B (and the edges that connect to B):



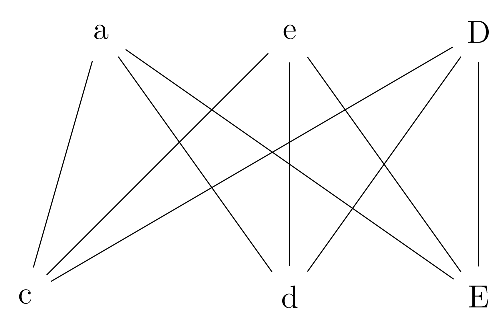
This subgraph is homeomorphic to $K_{3,3}$. To see why, first notice that the node b is just subdividing the edge from d to e, so we can delete it. Or, formally, the previous graph is a subdivision of this graph:



In the same way, we can remove the nodes A and C, to eliminate unnecessary subdivisions:



Now deform the picture a bit and we see that we have $K_{3,3}$.

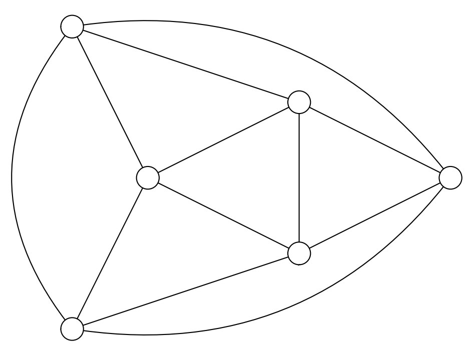


5 Application: Platonic solids

A fact dating back to the Greeks is that there are only five *Platonic solids*. These are convex polyhedra whose faces all have the same number of sides (k) and whose vertices all have the same number of edges going into them (d).

Show a picture of the five Platonic solids from the web: cube, dodecahedron, tetrahedron, icosahedron, octahedron, e.g. wikipedia "Platonic solids".

To turn a Platonic solid into a graph, imagine that it's made of a stretchy material. Make a small hole in one face. Put your fingers into that face and pull sideways, stretching that face really big and making the whole thing flat. For example, an octahedron (8 triangular sides) turns into the following graph. Notice that it still has eight regions, one for each face of the original solid, each with three sides.



Graphs of polyhedra are slightly special planar graphs. Polyhedra aren't allowed to have extra vertices partway along edges, so each vertex in the graph must have degree at least three. Also, since the faces must be flat and the edges straight, each face needs to be bounded by at least three edges.

So, if G is the graph of a Platonic solid, all the vertices of G must have the same degree $d \geq 3$ and all faces must have the same degree $k \geq 3$. I claim that the graphs of the five Platonic solids are the only planar graphs which satisfy these conditions.

Proof: By the handshaking theorem, the sum of the vertex degrees is twice the number of edges. So, since the degrees are equal to d, we have

$$dv = 2e$$

By the handshaking theorem for faces, the sum of the region degrees is also twice the number of edges. That is

$$kf = 2e$$

So this means that $v = \frac{2e}{d}$ and $f = \frac{2e}{k}$

Euler's formula says that $v - e + f = 2$. Substituting into this, we get:

$$\frac{2e}{d} - e + \frac{2e}{k} = 2$$

So

$$\frac{2e}{d} + \frac{2e}{k} = 2 + e$$

Dividing both sides by $2e$:

$$\frac{1}{d} + \frac{1}{k} = \frac{1}{e} + \frac{1}{2}$$

If we analyze this equation, we discover that d and k can't both be larger than 3. If they are both 4 or above, the left side of the equation is at most $\frac{1}{2}$. But since e is positive, the righthand side of the equation must be larger than $\frac{1}{2}$. So one of d and k is actually equal to three and the other is some integer that is at least 3.

Suppose we set d to be 3. Then the equation becomes

$$\frac{1}{3} + \frac{1}{k} = \frac{1}{e} + \frac{1}{2}$$

So

$$\frac{1}{k} = \frac{1}{e} + \frac{1}{6}$$

Since $\frac{1}{e}$ is positive, this means that k can't be any larger than 5.

Similarly, if k is 3, then d can't be any larger than 5.

This leaves us only five possibilities for the degrees d and k: $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$.

Once we've pinned down the degrees of all the vertices in the graph, we've pinned down the basic structure of the graph and of the corresponding solid figure. So there are only five possible graph structures and thus five possible Platonic solids.

At several points in this proof, it's probably not obvious why you would make that step e.g. in the algebra. This is the kind of proof that would have been constructed by trying several ideas and fiddling around with the algebra and the real-world geometrical problem. It's the kind of thing mathematicians do when stuck in the back of a boring committee meeting.