# Planar Graphs I

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This is a half-lecture due to the third quiz.

This lecture surveys facts about graphs that can be drawn in the plane without any edges crossing (first half of section 9.7 of Rosen).

## 1 Planar graphs

So far, we've been looking at general properties of graphs and very general classes of relations. Today, we'll concentrate on a limited class of graph: simple undirected connected graphs. Recall that a simple graph contains no self-loops or multi-edges. and connected means that there's a path between any two vertices. And we assume (without ever saying this explicitly) that all graphs are finite.

Which of these graphs are "planar" i.e. can be drawn in the plane without any edges crossing (i.e. not at a vertex)?

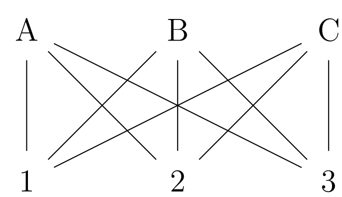
Examples: $K_4$ is planar, cube ($Q_3$) is planar, $K_{3,3}$ isn't. See pictures in Rosen p. 658.

Notice that some pictures of a planar graph may have crossing edges. What makes it planar is that you can draw at least one picture of the graph with no crossings.

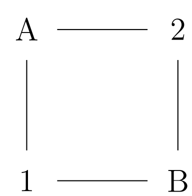
Why should we care? Connected to a variety of neat results in mathematics. (I'll show one Friday.) Also, crossings are a nuisance in practical design problems for circuits, subways, utility lines. Two crossing connections normally means that the edges must be run at different heights. This isn't a big issue for electrical wires, but it creates extra expense for some types of lines e.g. burying one subway tunnel under another (and therefore deeper than you would ordinarily need). Circuits, in particular, are easier to manufacture if their connections live in fewer layers.

## 2 $K_{3,3}$ isn't planar

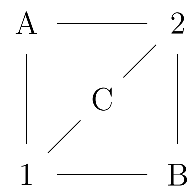
I just claimed that $K3, 3$ isn't planar. Let's see why this is really true. First, let's label the vertices:




The four vertices $A$, $B$, 1, and 2 form a cycle.




So $C$ must live inside the cycle or outside the cycle. Let's suppose it lives inside. (The argument is similar if it lives outside.) Our partial graph then looks like:




The final vertex 3 must go into one of the three regions in this diagram. And it's supposed to connect to $A$, $B$, and $C$. But none of the three regions has all three of these vertices on its boundary. So we can't add $C$ and its connections without a crossing.

This proof is ok, but it requires some care to make it convincing. Moreover, it's not going to generalize easily to more complex examples. So we're going to work out (today and Friday) some algebraic properties of planar graphs. This will let us prove that certain graphs aren't planar.

## 3 Faces

A planar graph divides the plane into a set of regions, also called *faces*. Each region is bounded by a simple cycle of the graph: the path bounding each region starts and ends at the same vertex and uses each edge only once. The numer of edges in this boundary is the *degree* of the face. By convention, we also count the unbounded area outside the whole graph as one region.

Examples: a cycle (2 regions), a figure 8 graph (3 regions), two nodes connected by a single edge (1 region).

This neat division of the plane into a set of regions seems intuitively obvious, but actually depends on a result from topology called the "Jordan curve theorem" which states that any simple closed curve (i.e. doesn't cross itself, starts and ends at the same place) divides the plane into exactly two regions. Proving this theorem requires worrying about the possibility that the curve has infinitely complex patterns of maze-like wiggles, but we won't go there.

Since planar graphs are more tightly constrained than general simple graphs, we have two basic formulas beyond our normal handshaking theorem. Specifically, if $e$ is the number of edges, $v$ is the number of vertices, and $f$ is the number of faces/regions, then

- Euler's formula says that $v - e + f = 2$.
- Handshaking theorm: sum of vertex degrees is $2e$
- Handshaking theorem for faces: sum of the face degrees is also $2e$.

To see why the handshaking theorem for faces holds, notice that each edge normally forms part of the boundary of two faces, one to each side of it. The few exceptions involve cases where the edge appears twice as we walk around the boundary of a single face. We'll prove Euler's formula below.

# 4 Trees

Before we try to prove Euler's formula, let's look at one special type of planar graph: trees. In graph theory, a tree is any connected graph with no cycles. When we normally think of a tree, it has a designated root (top) vertex. In graph theory, these are called *rooted trees*. For what we're doing this class, we don't need to care about which vertex is the root.

A tree doesn't divide the plane into multiple regions, because it doesn't contain any cycles. In graph theory jargon, a tree has only one face: the entire plane surrounding it. So Euler's theorem reduces to $v - e = 1$, i.e. $e = v - 1$. Let's prove that this is true, by induction.

> Proof by induction on the number of vertices in the graph.
>
> Base: If the graph contains no edges and only a single vertex, the formula is clearly true.
>
> Induction: Suppose the formula works for all trees with up to $n$ vertices. Let $T$ be a tree with $n + 1$ vertices. We need to show that $T$ has $n$ edges.

Now, we find a vertex with degree 1 (only one edge going into it). To do this start at any vertex $r$ and follow a path in any direction, without repeating edges. Because $T$ has no cycles, this path can't return to any vertex it has already visited. So it must eventually hit a dead end: the vertex at the end must have degree 1. Call it $p$.

Remove $p$ and the edge coming into it, making a new tree $T'$ with $n$ vertices. By the inductive hypothesis, $T'$ has $n-1$ edges. Since $T$ has one more edge than $T'$, $T$ has $n$ edges. Therefore our formula holds for $T$.

# 5 Proof of Euler's formula

We can now prove Euler's formula ($v - e + f = 2$) works in general, for any connected planar graph.

Proof: by induction on the number of edges in the graph.

Base: If $e = 0$, the graph consists of a single vertex with a single region surrounding it. So we have $1 - 0 + 1 = 2$ which is clearly right.

Induction: Suppose the formula works for all graphs with no more than $n$ edges. Let $G$ be a graph with $n+1$ edges.

Case 1: $G$ doesn't contain a cycle. So $G$ is a tree and we already know the formula works for trees.

Case 2: $G$ contains at least one cycle. Pick an edge $p$ that's on a cycle. Remove $p$ to create a new graph $G'$.

Since the cycle separates the plane into two regions, the regions to either side of $p$ must be distinct. When we remove the edge $p$, we merge these two regions. So $G'$ has one fewer regions than $G$.

Since $G'$ has $n$ edges, the formula works for $G'$ by the induction hypothesis. That is $v' - e' + f' = 2$. But $v' = v$, $e' = e - 1$, and $f' = f - 1$. Substituting, we find that

$$v - (e - 1) + (f - 1) = 2$$

So

$$v - e + f = 2$$