

Equivalence Relations and Partitions

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This lecture continues the topic of equivalence relations (section 8.5 of Rosen), including how to prove that something is an equivalence relation.

1 Announcements

Reminder: quiz next Wednesday.

2 Partitions

Equivalence relations are used to divide up a set A into equivalence classes, each of which can then be treated as a single object. E.g. all the equivalence fractions are treated as a single rational number. This only works properly if the equivalence classes divide up the set neatly: they cover the whole set A , they don't overlap, and every equivalence class contains at least one representative element. Cutting up A neatly is called being a *partition* of A .

Knowing that we have a partition means that we don't have to worry about how to handle annoying situations like partial overlap between two equivalence classes. This is particularly important when defining operations such as addition (e.g. of rationals). How would we define addition if one of the input equivalence classes had nothing in it?

Formally, a partition of a set A is a collection of non-empty subsets of

A which cover all of A and don't overlap. Specifically, if the subsets are A_1, A_2, \dots, A_n , then they must satisfy three conditions:

- a) $A_1 \cup A_2 \cup \dots \cup A_n = A$
- b) $A_i \neq \emptyset$ for all i
- c) $A_i \cap A_j = \emptyset$ for all $i \neq j$.

A partition can contain an infinite set of subsets. To cover this possibility, we need to use a more general notation. Let P be our partition. then the three conditions are:

- a) $\bigcup_{X \in P} X = A$
- b) $X \neq \emptyset$ for all $X \in P$
- c) $X \cap Y = \emptyset$ for all $X, Y \in P, X \neq Y$

3 Need for the RST properties

Any partition P has a corresponding equivalence relation. Specifically, we define $x \sim y$ if and only if x and y are in the same element of P . This relation is obviously reflexive, symmetric, and transitive.

Less obviously, if we start with any equivalence relation on A , its equivalence classes form a partition of A . This depends on the defining properties of an equivalence relation. Although we could apply the definition of an equivalence classes to any random relation, the sets it generates might not form a partition.

Suppose (for example) that our relation wasn't reflexive. Then $[x]$ wouldn't necessarily contain x . In fact, it's possible that $[x]$ might not contain any elements and/or x might not be in any equivalence class.

Suppose our relation wasn't symmetric. Then y might be in $[x]$ but x not in $[y]$. So $[x]$ and $[y]$ could intersect partially, without being the same set, and it might matter which representative element we picked to name

each equivalence class. This would make equivalence classes harder to use for mathematical or practical purposes.

Relations that aren't transitive also fail. Consider the relation R on the integers defined by xRy if and only if $|x - y| \leq 2$. Then $[3] = \{1, 2, 3, 4, 5\}$ and $[5] = \{3, 4, 5, 6, 7\}$. So, again, our subsets can intersect partially.

4 RST implies partition

Let's prove that if R is an equivalence relation on a set A , the equivalence classes of R form a partition of A . First, recall the precise definition of an equivalence class.

$$[x]_R = \{y \in A \mid xRy\}$$

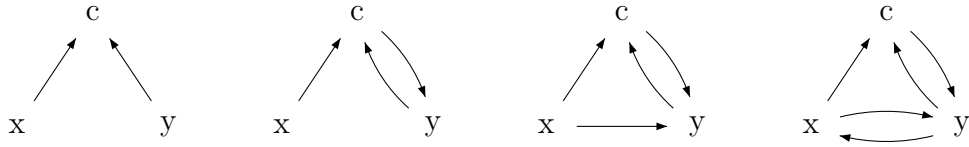
We could have written this definition using yRx in place of xRy . However, it's important that we consistently stick to one variant of the definition for the length of this proof.

First, since R is reflexive, xRx for any $x \in A$. So $x \in [x]$ for any $x \in A$. Therefore, no equivalence class is empty and the union of all equivalence classes is the whole set A . So the only thing that remains to be shown is that two distinct equivalence classes don't overlap.

Let x and y be two elements of A and suppose that $[x] \cap [y] \neq \emptyset$. We need to show that $[x] = [y]$.

Since $[x] \cap [y] \neq \emptyset$, we can pick an element c that is in $[x] \cap [y]$. I.e. $c \in [x]$ and $c \in [y]$. By the definition of equivalence class, this means that xRc and yRc . Since R is symmetric, we also have that cRy . So, by transitivity, xRy . And, thus by symmetry, yRx .

Here's a sequence of pictures showing which relationships we know at each step in this chain of reasoning. Notice that these pictures don't show all the edges in the relation, just the ones we've verified must be there. For example, cRx is missing even in the final diagram, because we never need to explicitly discuss the fact that this relationship also holds.



To show that $[x] \subseteq [y]$, pick any element $d \in [x]$. The definition of $[x]$ implies that xRd . Since we know that yRx , transitivity implies that yRd . So $d \in [y]$. A similar argument shows that $[y] \subseteq [x]$. So $[x] = [y]$.

5 Proving that a relation is an equivalence relation

So far, I've assumed that we can just look at a relation and decide if it's an equivalence relation. Suppose someone asks you (e.g. on an exam) to prove that something is an equivalence relation. These proofs just use techniques you've seen before. Let's do a couple examples.

First, remember that we had defined the set of fractions F to contain all fractions $\frac{x}{y}$ where $x, y \in \mathbb{Z}$ and $y \neq 0$. We constructed the rational numbers from the fractions by defining certain pairs of fractions to be equivalent. Specifically, our equivalence relation \sim was defined by: $\frac{x}{y} \sim \frac{p}{q}$ if and only if $xq = yp$.

Let's show that \sim is an equivalence relation.

Proof: Reflexive: For any x and y , $xy = xy$. So the definition of \sim implies that $\frac{x}{y} \sim \frac{x}{y}$.

Symmetric: if $\frac{x}{y} \sim \frac{p}{q}$ then $xq = yp$, so $yp = xq$, so $py = qx$, which implies that $\frac{p}{q} \sim \frac{x}{y}$.

Transitive: Suppose that $\frac{x}{y} \sim \frac{p}{q}$ and $\frac{p}{q} \sim \frac{s}{t}$. By the definition of \sim , $xq = yp$ and $pt = qs$. So $xqt = ypt$ and $pty = qsy$. Since $ypt = pty$, this means that $xqt = qsy$. Cancelling out the q 's, we get $xt = sy$. By the definition of \sim , this means that $\frac{x}{y} \sim \frac{s}{t}$.

Since \sim is reflexive, symmetric, and transitive, it is an equivalence relation.

Notice that the proof has three sub-parts, each showing one of the key properties. Each part involves using the definition of the relation, plus a small amount of basic math. The reflexive case is very short. The symmetric case is often short as well. Most of the work is in the transitive case.

There are relations for which one or the other property is hard to prove. Sometimes even reflexive can require some work. But it's more typical for these proofs to be almost mechanical.

6 Proving antisymmetry

Here's another example of a relation, this time an order (not an equivalence) relation. Consider the set of intervals on the real line $J = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$. Define the containment relation C as follows:

$$(a, b) C (c, d) \text{ if and only if } a \leq c \text{ and } d \leq b$$

Let's show that C is antisymmetric. For proofs, it's typically easiest to use this form of the definition of antisymmetry: if xRy and yRx , then $x = y$.

Notice that C is a relation on intervals, i.e. pairs of numbers, not single numbers. If we substitute the definition of C into the definition of antisymmetric, what we need to show is that $(a, b) C (c, d)$ and $(c, d) C (a, b)$ implies that $(a, b) = (c, d)$.

So, suppose that we have two intervals (a, b) and (c, d) such that $(a, b) C (c, d)$ and $(c, d) C (a, b)$. By the definition of C , $(a, b) C (c, d)$ implies that $a \leq c$ and $d \leq b$. Similarly, $(c, d) C (a, b)$ implies that $c \leq a$ and $b \leq d$.

Since $a \leq c$ and $c \leq a$, $a = c$. Since $d \leq b$ and $b \leq d$, $b = d$. So $(a, b) = (c, d)$.