Equivalence Relations

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21 April 2010

This lecture finishes the discussion of basic properties of relations and then covers material on equivalence classes from section 8.5 of Rosen, a topic we'll continue through next Monday.

1 Announcements

Reminder: quiz next Wednesday (28th).

There will be a makeup quiz on the last day of classes. It is on a mixture of the topics from quiz 2 and quiz 2 and you get 80% of the face value of the quiz. It's only worth taking if you missed a quiz (and it wasn't excused) or you have a truly awful score (e.g. ¡ 10) on one quiz.

Check your final exam schedule for conflicts. The final exam rules require that you report any conflicts to the instructors involved by the last day of classes.

There's two clarifications on HW 10, posted on the web page.

2 Transitive

The final important property of relations is transitivity. A relation R on a set A is *transitive* if transitive: for all $a, b, c \in A, aRb$ and bRc implies that aRc

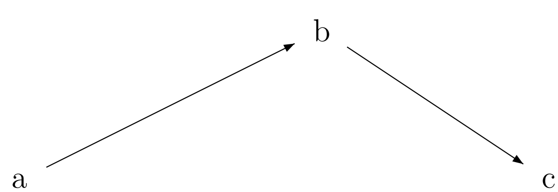
You've probably seen transitivity before, because it holds for a broad range of familiar numerical relations such as $<$, $=$, divides, and set inclusion. For example, for real numbers, if $x < y$ and $y < z$, then $x < z$. Similarly, if $x|y$ and $y|z$, then $x|z$. For sets, $X \subseteq Y$ and $Y \subseteq Z$ implies that $X \subseteq Z$.

If we look at graph pictures, transitivity means that whenever there is a path from x to y then there must be a direct arrow from x to y. This is true for S and B above, but not for W or Q.

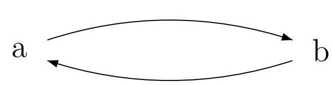
We can also understand this by spelling out what it means for a relation R on a set A not to be transitive:

not transitive: there are $a, b, c \in A, aRb$ and bRc and $a \not R c$

So, to show that a relation is not transitive, we need to find one counter-example, i.e. specific elements a, b, and c such that aRb and bRc but not aRc. In the graph of a non-transitive relation, you can find a subsection that looks like:

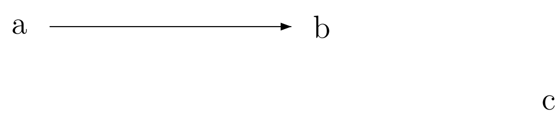


It could be that a and c are actually the same element, in which case the offending subgraph might look like:



The problem here is that if aRb and bRa, then transitivity would imply that aRa and bRb.

One subtle point about transitive is that it's an if/then statement. So it's ok if some sets of elements just aren't connected at all. For example, this subgraph is consistent with the relation being transitive.



A disgustingly counter-intuitive special case is the relation $P = \emptyset$ on any non-empty set, i.e. the relation in which no elements are related to one another. It's transitive, because it's never possible to satisfy the hypothesis of the definition of transitive. It's also symmetric, for the same reason. And, oddly enough, antisymmetric.

This special-case relation is irreflexive and not reflexive. Unlike symmetry and transitivity, reflexivity unconditionally requires that pairs of the form (x, x) must be in the relation. In this case, they aren't.

3 Types of relations

Now that we have these basic properties defined, we can define three important classes of relations:

- An equivalence relation is a relation that is reflexive, symmetric, and transitive.
- A partial order is a relation that is reflexive, antisymmetric, and transitive.
- A strict partial order is a relation that is irreflexive, antisymmetric, and transitive.

Equivalence relations act like equality, partial orders act like $\leq$ or $\geq$, and strict partial orders act like $<$ or $>$. In the picture examples above, S is an equivalence relation and T is a strict partial order. If we pick some collection of sets, then the set inclusion relation $\subseteq$ (no picture) would be a (non-strict) partial order on them.