Relations

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This lecture introduces relations and covers basic properties of relations, i.e. parts of section 8.1 and 8.3 of Rosen. When you look at Rosen, be aware that we're covering only relations on a single set, where he also covers relations between two sets.

1 Announcements

Third honors homework is due this Wednesday (21st).

Quiz 3 is coming up next Wednesday (28th).

2 More Connectivity

Before I forget, notice that the vertices in a graph are also often called "nodes."

Recall that, in an undirected graph, a path of length k from vertex a to vertex b is a sequence of edges that connect end-to-end

$$(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)$$

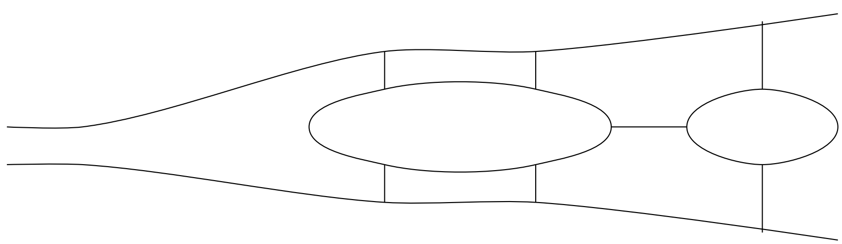
, where $v_1 = a$, $v_k = b$.

An undirected graph G is connected if there is a path between every pair of vertices in G. That is, for any vertices a and b in G, there is a path from a to b.

Three special types of paths are important to know about

- A circuit is a path that ends at the same vertex where it started.
- A path is simple if no edge occurs more than once in the path.
- An Euler circuit of a graph G is a simple circuit that contains every edge in the graph.

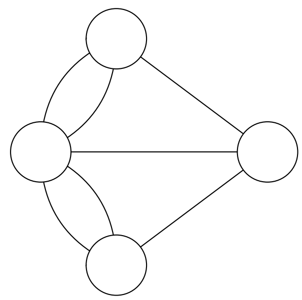
Fascination with Euler circuits dates back to the 18th century. At that time, the city of Könisgberg, in Prussia, had a set of bridges that looked roughly as follows:



Folks in the town wondered whether it was possible to take a walk in which you crossed each bridge exactly once, coming back to the same place you started. This is the kind of thing that starts long debates late at night in pubs, or keeps people amused during boring church services. Leonard Euler was the one who explained clearly why this isn't possible.

Specifically, an Euler circuit is possible exactly when each vertex has even degree. In order to complete the circuit, you have to leave each vertex that you enter. If the vertex has odd degree, you will eventually enter a vertex but have no unused edge to go out on.

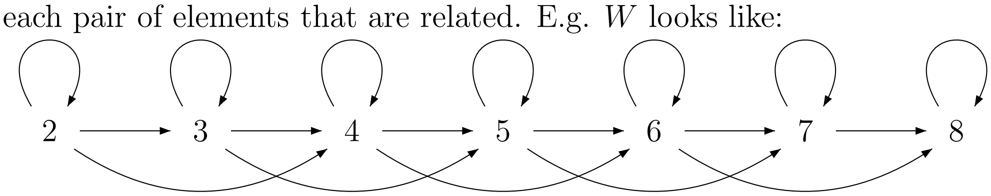
For our specific example, the corresponding graph looks as follows. Since all of the vertices have odd degree, there's no possibility of an Euler circuit.



3 Relations

A *relation R on a set A* is a subset of $A \times A$, i.e. R is a set of ordered pairs of elements from A. If R contains the pair (x, y), we say that x is related to y, or xRy in shorthand.[1] We'll write $x \not R y$ to mean that x is not related to y.

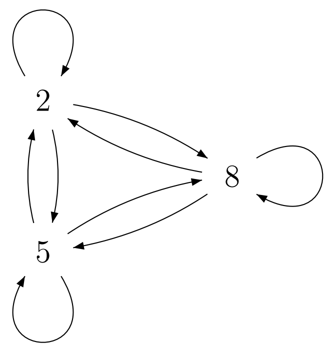
For example, suppose we let $A = \{2, 3, 4, 5, 6, 7, 8\}$. We can define a relation W on A by xWy if and only if $x \leq y \leq x + 2$. Then W contains pairs like $(3, 4)$ and $(4, 6)$ and $(5, 5)$, but not the pairs $(6, 4)$ and $(3, 6)$. We can draw pictures of relations using directed graphs, with an arrow joining each pair of elements that are related. E.g. W looks like:



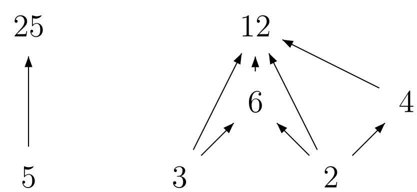
In fact, there's very little formal difference between a relation on a set A and a directed graph, because graph edges can be represented as ordered pairs of endpoints. They are two ways of describing the same situation.

We can define another relation S on A by saying that xSy is in S if $x \equiv y \pmod 3$. Then S would look like:

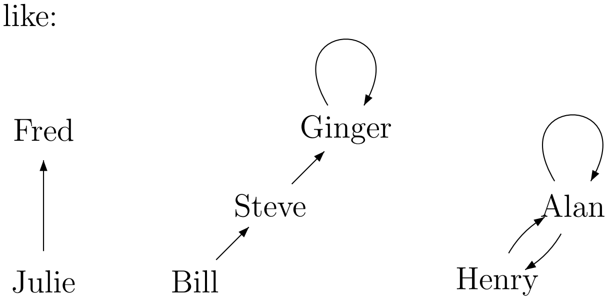
[1]The textbook writes $(x, y) \in R$ for the first few sections and then switches to this shorthand. I find the shorthand much easier to read.



Or, suppose that $B = \{2, 3, 4, 5, 6, 12, 25\}$. Let's set up a relation T on B such that xTy if $x|y$ and $x \neq y$. Then our picture would look like



Mathematical relations can also be used to represent real-world relationships, in which case they often have a less regular structure. For example, suppose that we have a set of students and student x is related to student y if x nominated y for ACM president. The graph of this relation (call it Q) might look like:



Relations can also be defined on infinite sets or multi-dimensional objects. For example, we can define a relation Z on the real plane $\mathbb{R}^2$ in which (x, y) is related to (p, q) if and only if $x^2 + y^2 = p^2 + q^2$. In other words, two points are related if they are the same distance from the origin.

For complex relations, the full directed graph picture can get a bit messy. So there are simplified types of diagrams for certain specific special types of relations, e.g. the so-called Hasse diagram for partial orders.

4 Properties of relations: reflexive

Relations are classified by several key properties. The first is whether an element of the set is related to itself or not. There are three cases

- Reflexive: every element is related to itself.
- Irreflexive: no element is related to itself.
- Neither reflexive nor irreflexive: some elements are related to themselves but some aren't.

In our pictures above, elements related to themselves have self-loops. So it's easy to see that W and S are reflexive, T is irreflexive, and Q is neither. The familiar relations $\leq$ and $=$ on the real numbers are reflexive, but $<$ is irreflexive. Suppose we define a relation M on the integers by xMy if and only if $x + y = 0$. Then 2 isn't related to itself, but 0 is.

The formal definition states that if R is a relation on a set A then

- R is reflexive if xRx for all $x \in A$.
- R is irreflexive if $x \not R x$ for all $x \in A$.

Notice that irreflexive is not the negation of reflexive. The negation of reflexive would be:

- not reflexive: there is an $x \in A, x \not R y$

5 Symmetric and antisymmetric

Another important property of a relation is whether the order matters within each pair. That is, if (x, y) is in R, is (y, x) always in R? A relation satisfying this property is called *symmetric*. In a graph picture of a symmetric relation, a pair of elements is either joined by a pair of arrows going in opposite directions, or no arrows. In our examples with pictures above, only S is symmetric.

Relations that resemble equality are normally symmetric. For example, the relation X on the integers defined by xXy iff $|x| = |y|$ is symmetric. So is the relation N on the real plane defined by $(x, y)N(p, q))$ iff $(x-p)^2 + (y-q)^2 \leq 25$ (i.e. the two points are no more than 5 units apart).

Relations that put elements into an order, like $\leq$ or divides, have a different property called *antisymmetry*. A relation is *antisymmetric* if two distinct elements are never related in both directions. In a graph picture, a pair of points may be joined by a single arrow, or not joined at all. In our pictures above, W and T are antisymmetric.

As with reflexivity, there are mixed relations that have neither property. So the relation Q above is neither symmetric nor antisymmetric.

If R is a relation on a set A, here's the formal definition of what it means for R to be symmetric (which doesn't contain anything particularly difficult):

symmetric: for all $x, y \in A, xRy$ implies yRx

There's two ways to define antisymmetric. They are logically equivalent and you can pick whichever is more convenient for your purposes:

antisymmetric: for all x and y in A with $x \neq y$, xRy implies $y \not R x$

antisymmetric: for all x and y in A, xRy and yRx implies $x = y$

To interpret the second definition, remember that when mathematicians pick two values x and y, they leave open the possibility that the two values are actually the same. If we said that in normal conversational English, we would normally mean that they had to be different. I find that the first definition is better for understanding the idea of antisymmetry, but the second is more useful for writing proofs.